CHAPTER I: MATRICES and VECTORS

Every book should have a unifying theme. For this book, that theme will be **Matrices**. This chapter will only scratch the surface, presenting enough terminology to introduce, in all subsequent chapters, deep and potentially confusing ideas intuitively and compactly.

SECTION IA: MATRICES, EXAMPLES, and ALGERA.

DEFINITION 1.1

A **matrix** is a rectangular array of real numbers.

More specifically, for n, m natural numbers, an **(m×n)** (reads "m by n") **matrix**

is a rectangular array of m rows (horizontal sequences of n real numbers) and n columns (vertical sequences of m real numbers)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

a_{ij} denotes the ij^{th} entry, the number in the i^{th} row and j^{th} column, for $1 \le i \le m$, $1 \le j \le n$.

Example 1.2

$$A = \begin{bmatrix} 1 & 0 & (-2) \\ 1 & 3 & 5 \end{bmatrix}$$

has 2 rows: $\begin{bmatrix} 1 & 0 & (-2) \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$

and 3 columns: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} (-2) \\ 5 \end{bmatrix}$

thus is a (2×3) matrix, with

$a_{11} = 1$, $a_{12} = 0$, $a_{13} = (-2)$, $a_{21} = 1$, $a_{22} = 3$, and $a_{23} = 5$.

DEFINITIONS 1.3

Addition and scalar multiplication of matrices is done entrywise:

$(a_{ij}) + (b_{ij}) \equiv ((a_{ij} + b_{ij}))$,

$c(a_{ij}) \equiv ((c\,a_{ij}))$,

for c real, (a_{ij}) and (b_{ij}) both $(m \times n)$ matrices.

Example 1.4

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 4 \end{bmatrix}$$

$$3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

Example 1.5

A matrix can be used to store information. Consider the following table, relating time per day spent on the Internet (small, medium, or large amounts of time) to grades received in a math class (E, D, C, B, or A).
For example, the table is stating that 15 people who spent a small amount of time on the Internet got a B.

		grade				
		E	D	C	B	A
time on Internet	small	2	3	10	15	45
	medium	8	7	10	20	25
	large	20	8	12	5	0

This table could be compactly written as a matrix

$$G \equiv \begin{bmatrix} 2 & 3 & 10 & 15 & 45 \\ 8 & 7 & 10 & 20 & 25 \\ 20 & 8 & 12 & 5 & 0 \end{bmatrix}$$

Notice the implicit moralizing in this matrix.

TERMINOLOGY 1.6

If A_k is an $(m \times n_k)$ matrix, for $1 \leq k \leq p$, then

$$[A_1 \; A_2 \; A_3 \cdots A_p] \equiv A$$

denotes the $(m \times (n_1 + n_2 + \cdots + n_p))$ matrix such that columns 1 thru n_1 are columns 1 thru n_1 of A_1, columns (n_1+1) thru n_2 are columns 1 thru n_2 of A_2, etc.

Define the $((m_1 + m_2 + \cdots m_p) \times n)$ matrix

$$A \equiv \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_p \end{bmatrix}$$

similarly,

where A_k is an $(m_k \times n)$ matrix, for $1 \le k \le p$.

Examples 1.7

If $A_1 \equiv \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $A_2 \equiv \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}$, and $A_3 \equiv \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, then

$$[A_1 \ A_2 \ A_3] \equiv \begin{bmatrix} 1 & 2 & 3 & 7 & 8 & -1 \\ 4 & 5 & 6 & 9 & 10 & -2 \end{bmatrix}$$

If $A_1 \equiv \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ and

$A_2 \equiv \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$, then

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \equiv \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

The technical term for Terminology 1.6 is

"pasting"

DEFINITIONS 1.8

A **row n-vector** is a $(1 \times n)$ matrix and a **column m-vector** is an $(m \times 1)$ matrix.

Examples 1.9

$\begin{bmatrix} 1 & 0 & -1 & \pi \end{bmatrix}$ is a row 4-vector

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a column 3-vector

DEFINITION 1.10

We may multiply a row n-vector (on the left) times a column n-vector:

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} =$$

$$\begin{bmatrix} (x_1 y_1 + x_2 y_2 + \cdots + x_n y_n) \end{bmatrix}.$$

Example 1.11

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1(-1) + 2\cdot 4 + 3\cdot 0 \end{bmatrix} = \begin{bmatrix} 7 \end{bmatrix}.$$

Notice the disappointment and loss of information:

Two n-vectors collapse into a number, or (1×1) matrix.

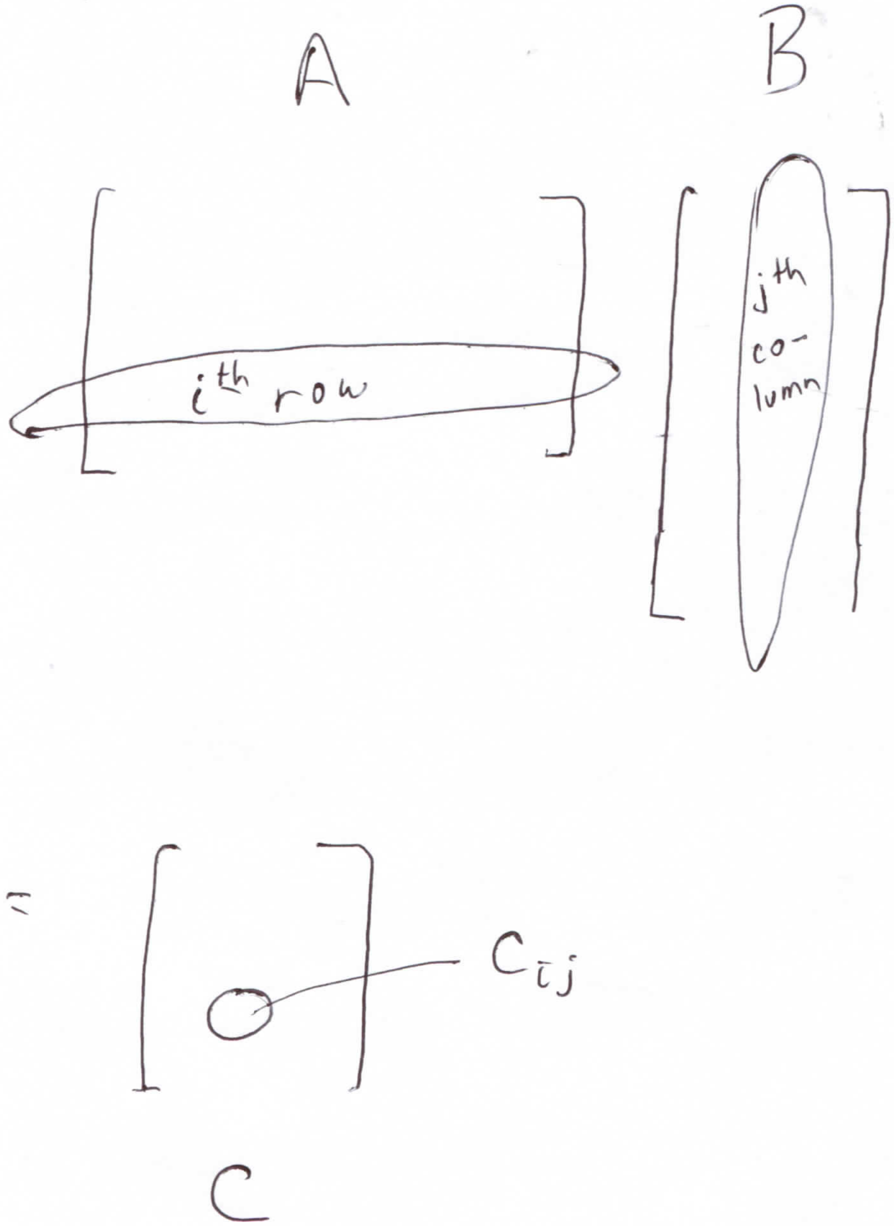
This will be called a **dot product** in Chapter VI.

DEFINITION 1.12

In general, matrix multiplication means doing all possible rows times columns.

Specifically, if $A = (a_{ij})$ is an $(m \times k)$ matrix and $B = (b_{ij})$ is a $(k \times n)$ matrix, then $C = (c_{ij}) = AB$ is an $(m \times n)$ matrix, with

$$c_{ij} = \left[i^{th} \text{ row of } A \right] \begin{bmatrix} j^{th} \\ \text{column} \\ \text{of } B \end{bmatrix}$$



$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ik} b_{kj}$$

Examples 1.13

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 + 2 + 3 \end{bmatrix} = \begin{bmatrix} 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 + 1 + 1 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

Example 1.14

Let G be as in Example 1.5.

Assume now that grades are subsidized: \$10 for each A, \$5 for each B, \$3 for each C, \$1 for each D, and nothing for an E.

Represent this money as a column 5-vector

$$\begin{bmatrix} E \\ D \\ C \\ B \\ A \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 5 \\ 10 \end{bmatrix}$$

When we multiply

$$G \begin{bmatrix} 0 \\ 1 \\ 3 \\ 5 \\ 10 \end{bmatrix},$$

we get a column 3-vector

$$\begin{bmatrix} \text{small} \\ \text{medium} \\ \text{large} \end{bmatrix} = \begin{bmatrix} 0 + 3 + 30 + 75 + 450 \\ 0 + 7 + 30 + 100 + 250 \\ 0 + 8 + 36 + 25 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 558 \\ 387 \\ 69 \end{bmatrix} = \begin{bmatrix} \text{money for ``small''} \\ \text{money for ``medium''} \\ \text{money for ``large''} \end{bmatrix}$$

The following properties of our matrix operations are tedious to prove.

PROPERTIES 1.15

In each of the following, A, B and C are matrices for which the indicated operations are defined and α and β are real numbers.

(1) and (7) are <u>commutative</u> properties; (2), (4), and (6) are <u>associative</u> properties; (3) and (5) are <u>distributive</u> properties.

1.15

(1) $A + B = B + A$

(2) $A + (B + C) = (A + B) + C$

(3) $(\alpha + \beta)A = \alpha A + \beta A$ and
$\alpha(A + B) = \alpha A + \alpha B$

(4) $\alpha(\beta A) = (\alpha \beta)A$

(5) $A(B + C) = AB + AC$ and
$(B + C)A = BA + CA$

(6) $A(BC) = (AB)C$

(7) $A(\alpha B) = \alpha(AB)$

REMARK 1.16

Please notice that matrix multiplication is <u>not</u> commutative.

For example, if $A \equiv \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B \equiv \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$,

then

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = BA$$

We will come back to this in Examples 1.23.

SECTION IB: VECTORS

DEFINITIONS 1.17

For n a natural number, an **n-vector** is an ordered n-tuple

$$(x_1, x_2, \ldots, x_n)$$

of real numbers, that is, a sequence of n real numbers.

Denote by

$\mathbb{R}^n$ (reads "R-enn")

$$\equiv \left\{ \vec{x} \equiv (x_1, x_2, \ldots, x_n) \,\middle|\, \begin{array}{l} x_j \text{ real,} \\ 1 \le j \le n \end{array} \right\}$$

For $1 \le j \le n$ x_j is the j^{th} **component** of $\vec{x}$

Example 1.18

$(7, 0, -\sqrt{2}, 5)$ is a 4-vector, with components $x_1 = 7$, $x_2 = 0$, $x_3 = -\sqrt{2}$, $x_4 = 5$.

REMARKS 1.19

We have already seen (Definitions 1.8) row n-vectors

$$[x_1 \ x_2 \ \ldots \ x_n]$$

and column n-vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

An "n-vector" does not distinguish between columns or rows or any other representation of n real numbers placed in order; arranging said numbers as rows or columns is done for (usually multiplicative) convenience.

The adjective "ordered" means order matters; for example,

$$(1,7) \neq (7,1),$$

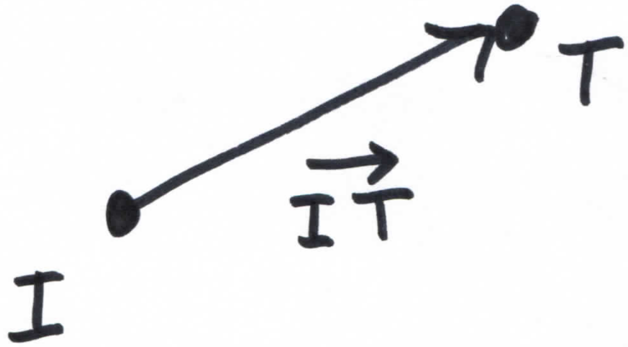
even though each vector has $\{1,7\}$ as $\{\text{components}\}$.

We now need to explain the arrow in the terminology $\vec{x}$ for a vector in $\mathbb{R}^n$.

DEFINITIONS 1.20

There is a very convenient picture of an n-vector, which can be given literal sense when $n = 2$.

We need <u>two</u> points in the Cartesian plane, call them I and T. The **directed line segment** $\overrightarrow{IT}$ is an arrow from I to T; that is, the tip of the arrow is at T and a fat dot, a nondecorative version of feathers, is at I.

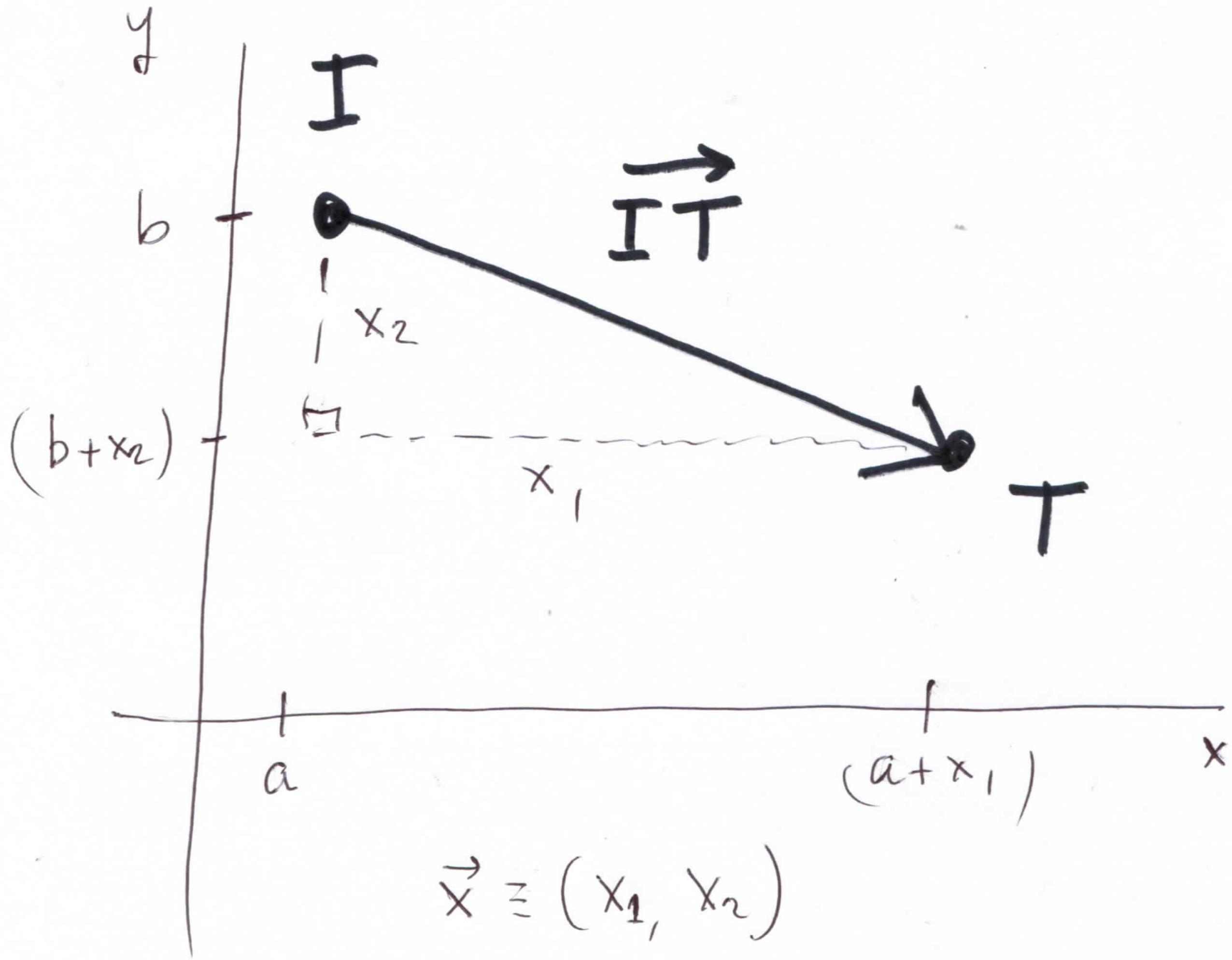


I is the <u>initial</u> <u>point</u> of $\overrightarrow{IT}$ and T is the

<u>**terminal**</u> <u>**point**</u>.

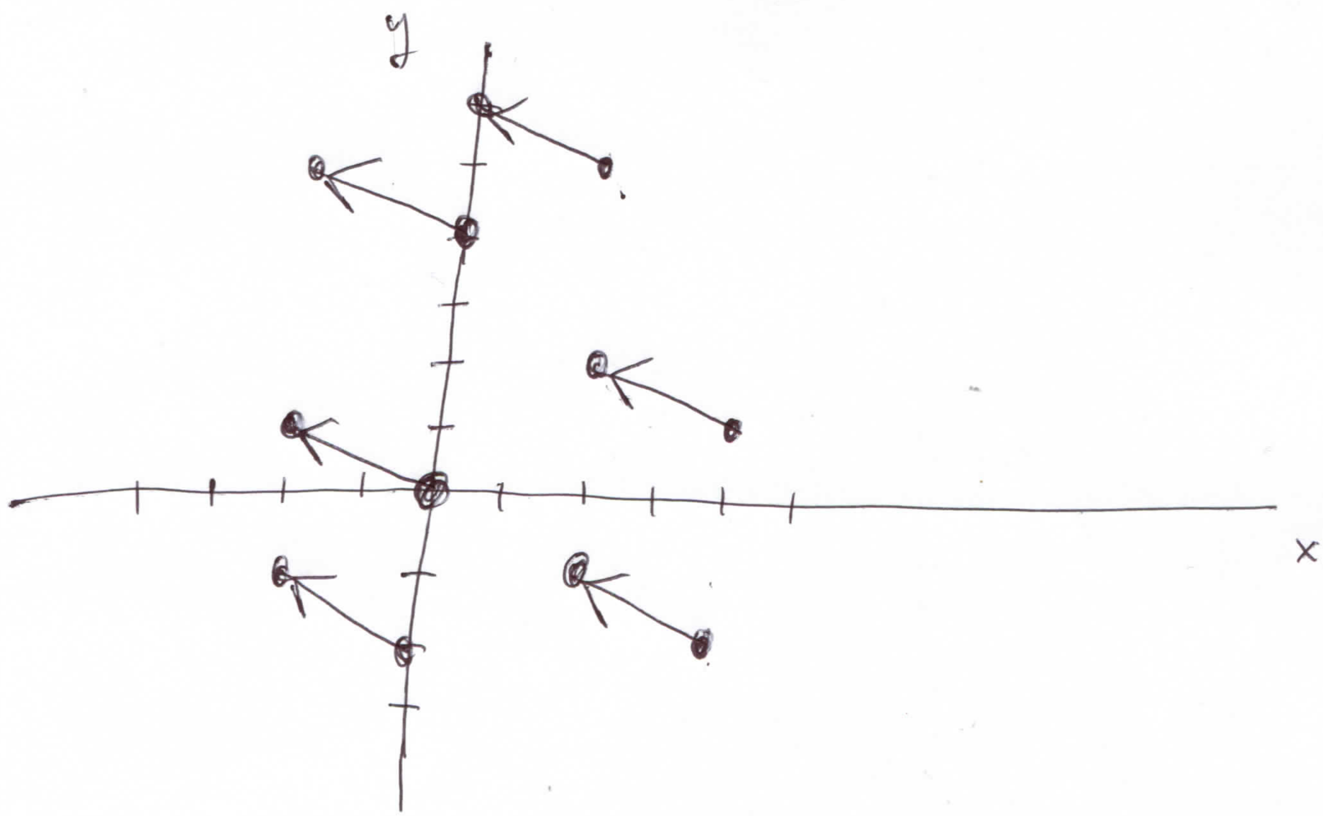
Notice the sense of motion, travelling from I to T; this is the "directed" part of "directed line segment."

The directed line segment $\overrightarrow{IT}$ **represents** the vector $\vec{x} \equiv (x_1, x_2)$ if, for some real a, b, $I = (a, b)$ and $T = (a + x_1, b + x_2)$.



Example 1.21

Each of the directed line segments drawn below represents the vector $(-2, 1)$.

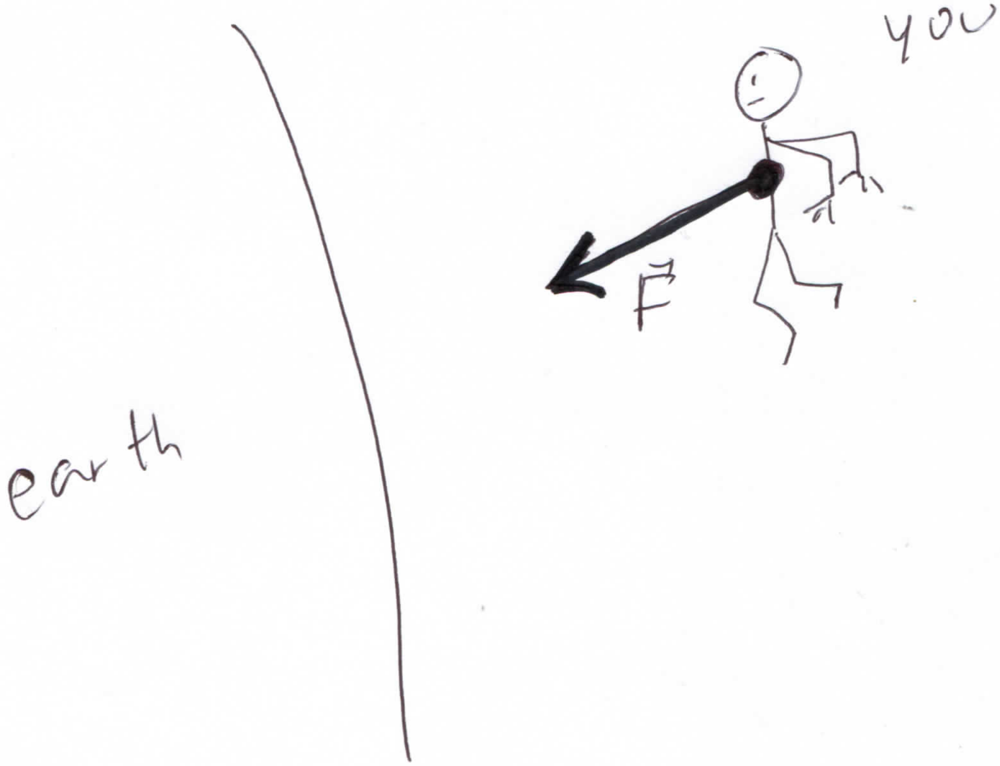


What matters is not where the arrow begins or ends, it is the relationship between the initial point and the terminal point; the vector is describing the <u>displacement</u> in moving from I to T.

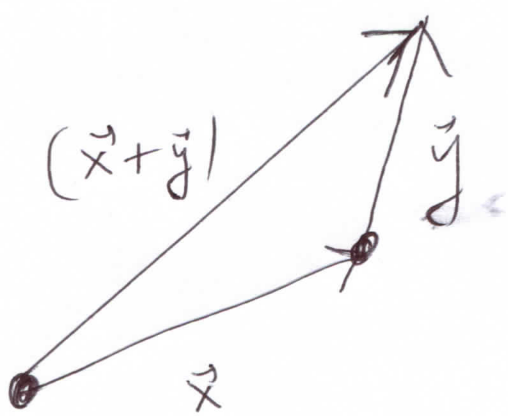
Intuitively, two directed line segments represent the same vector if they have the same length and direction.

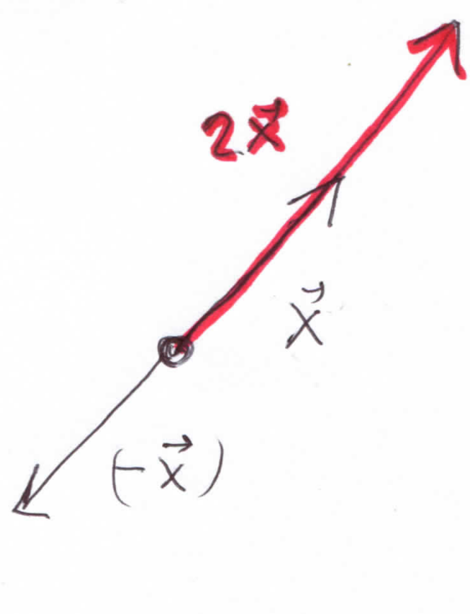
In physics, the first thing one needs to know about a parameter is whether it's a scalar (number) or a vector (magnitude and direction). For example, speed, e.g. 30 miles per hour, is a scalar, while velocity, e.g. 30 miles per hour north by northwest, is a vector.

Being able to move arrows around and still represent the same vector is very convenient. If you're in space (in a space suit), it is natural to place the force due to gravity starting at your chest; it is where you _feel_ the force, call it $\vec{F}$.



This flexibility regarding the initial point of a directed line segment gives the following pictures for the addition and scalar multiplication inherited by vectors from matrices (Definitions 1.3)

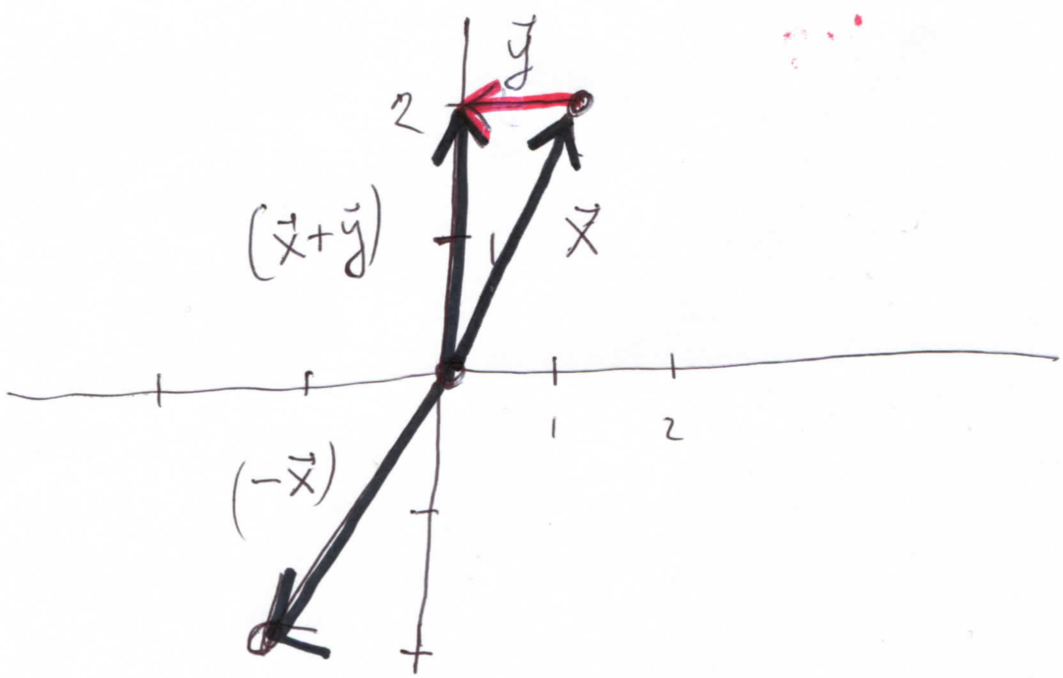




This picture is informative in $\mathbb{R}^n$, for any natural number n, even though it makes literal sense only for $n = 2$.

Examples 1.22

Draw $\vec{x} \equiv (1, 2)$, $\vec{y} \equiv (-1, 0)$, $(\vec{x} + \vec{y})$, and $(-\vec{x})$.



SECTION IC: DIFFERENCE EQUATIONS

Examples 1.23

We would like to think of matrix multiplication as "doing something" to vectors.

For example, if

$$A \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ then,}$$

for any real x_1, x_2,

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix},$$

that is, multiplication by A interchanges components, like a shell game.

Chapter VII will explore this aspect of matrices. For now, let's talk about the noncommuting in Remark 1.16.

Let A represent
"open the window,"
B represent "put head thru window".

You should believe (the experimental method is not recommended here) that

"A, followed by B"

is different than

"B, followed by A".

Order of operation makes a difference: AB often does not equal BA.

Often a matrix is applied repeatedly at regular intervals.

DEFINITION 1.24

A **Difference Equation** has the form

$$\vec{x}_{k+1} = A\vec{x}_k, \quad k = 0, 1, 2, \ldots$$

where, for some n, A is an $(n \times n)$ matrix and $\vec{x}_k$ is an n-vector, $k = 0, 1, 2, \ldots$

The number k should be thought of as time; e.g., $k \equiv$ number of hours after noon on January 1, 2017; or $k \equiv$ number of generation after predators were introduced into an ecosystem.

A **solution** of the Difference Equation is a sequence of n-vectors $\{\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots\}$ that satisfy the Difference Equation.

Examples 1.25

(1) $A \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ as above,

$\vec{x}_0 \equiv \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then

$$\vec{x}_1 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$\vec{x}_2 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$\vec{x}_3 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ldots$$

the solution of this Difference Equation is $\{\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots\}$

$$= \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ldots \right\}.$$

We seem to be doomed to alternate forever between $(1,2)$ (even terms) and $(2,1)$ (odd terms).

(2) $A \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\vec{x}_0 \equiv \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (again)

$$\vec{x}_1 \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

$$\vec{x}_2 \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \quad (= -\vec{x}_0)$$

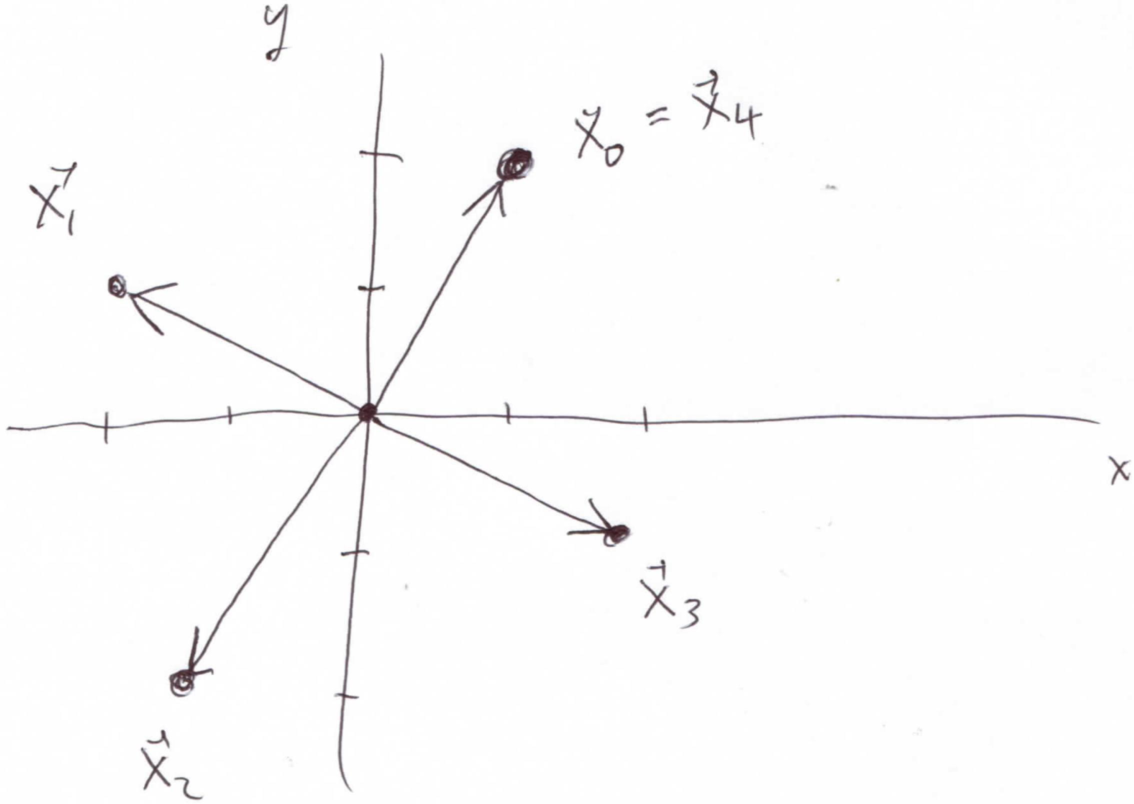
$$\vec{x}_3 \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad (= -\vec{x}_1)$$

$$\vec{x}_4 \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{x}_0, \ldots$$

As with (1), our destiny is sealed, with repetition every time k increases by 4:

$$\vec{x}_{k+4} = \vec{x}_k, \qquad k = 0, 1, 2, \ldots$$

A picture gives us a clue:



Consecutive vectors appear to form a right angle. For this reason,

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is called a <u>rotation matrix</u>;

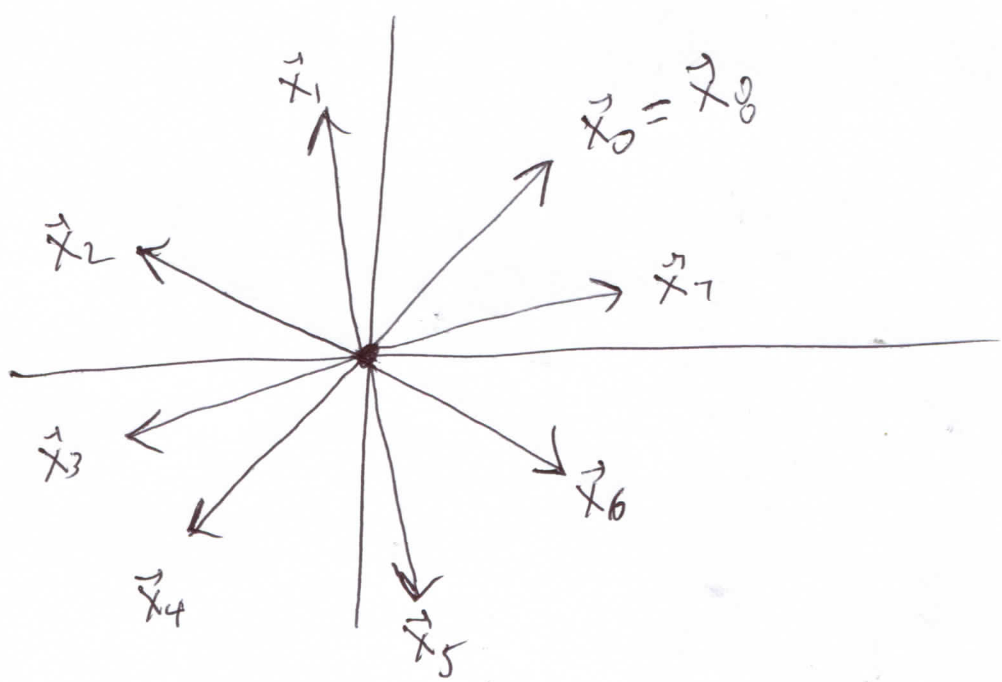
for those who have seen some trigonometry, these are discussed in Appendix One.

(3) $A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

turns out to be another rotation matrix, with, for any $\vec{x}_0$,

$$\vec{X}_{k+8} = \vec{X}_k, \quad k = 0, 1, 2, \ldots;$$

the solution to this Difference Equation will repeat every time k increases by 8;



(4) Let's talk about a predator-prey relationship, as in the Introduction.

Denote, for any $k = 0, 1, 2, 3, \ldots$

$r_k \equiv$ number of rabbits k years from now;

$f_k \equiv$ number of foxes k years from now.

We'll assume the following plausible activity each year:

$$(*)\begin{cases} f_{k+1} = 4 f_k \\ r_{k+1} = 100 r_k - 360 f_k \end{cases} \quad \begin{pmatrix} k=0,1, \\ 2,3,\ldots \end{pmatrix}$$

The "4" and "100" in $(*)$ represent reproduction; the fact that 100 is much bigger than 4 is an old joke. The "360" in $(*)$ is the number of rabbits eaten by each fox in a year; that's the "interaction".

We can write (*) as a Difference Equation:

For $k = 0, 1, 2, \ldots$, denote

$$\vec{x}_k \equiv \begin{bmatrix} r_k \\ f_k \end{bmatrix}, \quad \text{then}$$

$$\vec{x}_{k+1} = \begin{bmatrix} r_{k+1} \\ f_{k+1} \end{bmatrix} = \begin{bmatrix} 100 r_k - 360 f_k \\ 4 f_k \end{bmatrix}$$

$$= \begin{bmatrix} 100 & -360 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} r_k \\ f_k \end{bmatrix}$$

$$= \begin{bmatrix} 100 & -360 \\ 0 & 4 \end{bmatrix} \vec{x}_k \quad (k = 0, 1, 2, \ldots)$$

that is, we have the Difference Equation 1.24, with

$$A \equiv \begin{bmatrix} 100 & -360 \\ 0 & 4 \end{bmatrix}.$$

Actually, the model (*) breaks down when and if $r_k \leq 0$ (extinction).

Let's say we're starting with 12 foxes and 50 rabbits. We'd like to know future populations; in particular, we'd like to know if extinction occurs, and if so, when.

Well,

$$\vec{x}_0 = \begin{bmatrix} 50 \\ 12 \end{bmatrix}, \quad \text{so}$$

$$\vec{x}_1 = \begin{bmatrix} 100 & -360 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 50 \\ 12 \end{bmatrix} = \begin{bmatrix} 680 \\ 48 \end{bmatrix};$$

680 rabbits and 48 foxes next year.

$$\vec{x}_2 = \begin{bmatrix} 100 & -360 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 680 \\ 48 \end{bmatrix} = \begin{bmatrix} 50,720 \\ 192 \end{bmatrix};$$

50,720 rabbits and 192 foxes two years from now.

The fox populations are predictable (quadruples every year), but no pattern to the rabbit populations seem apparent.

We will come back to this model in Chapter VIII. Extinction turns out to depend entirely on $\frac{r_0}{f_0}$, the initial ratio of rabbit to fox populations. For our model (*), extinction occurs eventually if and only if

$$\frac{r_0}{f_0} < 3.75 \, ;$$

no extinction when we begin with 50 rabbits and 12 foxes, but beginning with 40 rabbits and 12 foxes would mean eventual extinction.

See Chapter VIII for details.

(5) Suppose Moonorgs are the only living entities on the moon and can also survive in space. Suppose also that, every year, 20% of Moonorgs on the moon leave the moon, while 10%

of Moonorgs not on the moon move to the moon.

If there are currently 50,000 Moonorgs on the moon and 10,000,000 Moonorgs not on the moon, and there are no births or deaths, how many Moonorgs will be on the moon in 3 years?

To set this up as a Difference Equation, let, for $k = 0, 1, 2, \ldots$,

$$\vec{x}_k \equiv \begin{bmatrix} \left(\text{number of Moonorgs not on the moon } k \text{ years from now}\right) \\ \left(\text{number of Moonorgs on the moon } k \text{ years from now}\right) \end{bmatrix}$$

Then $\vec{x}_0 = \begin{bmatrix} 10{,}000 \\ 50 \end{bmatrix}$ (in thousands)

and, if

$$A \equiv \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix},$$

it can be shown (see Chapter VII) that

$$\vec{x}_{k+1} = A\,\vec{x}_k \quad (k = 0, 1, 2, \ldots)$$

To get to the third year, at least in our present state of knowledge, we must pass through years one and two:

$$\vec{X}_1 = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 10,000 \\ 50 \end{bmatrix} = \begin{bmatrix} 9,010 \\ 1,040 \end{bmatrix}$$

$$\vec{X}_2 = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 9,010 \\ 1,040 \end{bmatrix} = \begin{bmatrix} 8,317 \\ 1,733 \end{bmatrix}$$

$$\vec{X}_3 = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 8,317 \\ 1,733 \end{bmatrix} = \begin{bmatrix} 7831.9 \\ 2218.1 \end{bmatrix}$$

Only the second component is asked for: 2,218.1 thousands or

$\boxed{\text{2,218,100 Moonorgs}}$

Notice, in the last example, that the sum of the components in $\vec{x}_0$, $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$ is the same ("no births or deaths" was assumed).

Notice also the matrix

$$A \equiv \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$$ that

does the population rearrangement has each of its columns adding up to 1, with all nonnegative entries.

DEFINITIONS 1.26

An $n \times n$ matrix A is a **Markov matrix** if all its entries are nonnegative and in each column the entries add up to one.

The corresponding Difference Equation

$$\vec{x}_{k+1} = A\vec{x}_k \quad (k = 0, 1, 2, \ldots)$$

is then called a

Markov process;

the sums of the components of $\vec{x}_k$ are then the same, for all k.

$$A = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix},$$

from the last example, is an example of a Markov matrix.

REMARKS 1.27

In any Difference Equation 1.24, the entire sequence $\{\vec{x}_k\}_{k=0}^{\infty}$ is uniquely determined by A and $\vec{x}_0$:

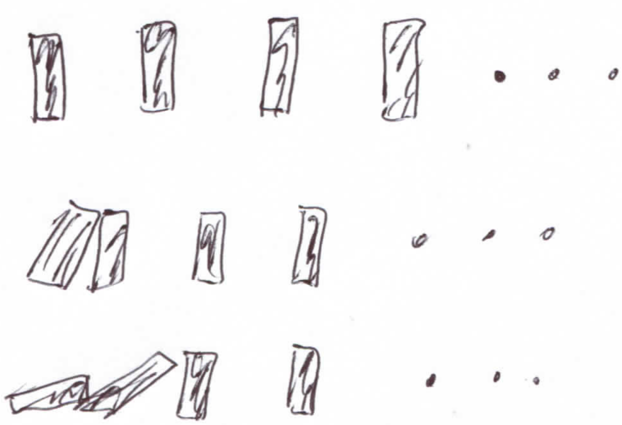
$\vec{x}_1 = A\vec{x}_0$,

$\vec{x}_2 = A\vec{x}_1 = AA\vec{x}_0$,

$\vec{x}_3 = A\vec{x}_2 = AAA\vec{x}_0$,

etc.;

$\vec{x}_0$, the **initial state**,
is like the first domino
in a line of dominoes;
knocking it over causes all
the dominoes to fall:



To describe $\vec{x}_k$,
for all k, explicitly,
we need some matrix definitions.

DEFINITIONS 1.28

An $(n \times n)$ matrix is called
a **square matrix**.

If A is a square matrix, then

$$A^2 \equiv AA, \quad A^3 \equiv A(A^2), \ldots$$

$$A^{k+1} \equiv A(A^k), \quad k = 1, 2, 3, \ldots$$

THEOREM 1.29

If $\{\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots\}$ is a solution of the Difference Equation 1.24, then

$$\vec{x}_n = A^n \vec{x}_0 \qquad n = 1, 2, 3 \ldots$$

REMARKS 1.30

The fact that we can predict the future $\vec{x}_1, \vec{x}_2, \ldots$ in a Difference Equation, just by knowing the initial state $\vec{x}_0$, borders on super powers. In our predator-prey model

(4) of Examples 1.25, just by wandering around counting (rabbit and fox) bodies, we can assert populations arbitrarily far into the future.

A serious disadvantage to Examples 1.25(4), and all Difference Equations, is that, to get to $\vec{x}_n$, you must perform n multiplications:

$\vec{x}_1 = A\vec{x}_0, \quad \vec{x}_2 = A\vec{x}_1, \ldots,$

$\vec{x}_n = A\vec{x}_{(n-1)}.$

Theorem 1.29 does not automatically help us, since calculating A^n is also $(n-1)$ multiplications:

$$A^2 = AA, \quad A^3 = A(A^2), \quad \ldots$$

$$A^n = A(A^{n-1}).$$

What we need is an explicit formula, involving only numbers and "n", for A^n. This will be accomplished in Ch. VIII, using all the tools we will develop between now and then.