CHAPTER III: QUALITATIVE INFORMATION

about linear systems and matrices

There are many reasons we might not be interested in explicit solutions of a linear system, difference equation, or other physical models associated with a matrix, e.g., uncertainty about the precise values of the entries in a matrix; but we might want qualitative information. Is a linear system consistent? If so, how many solutions does it have? In a predator-prey world, does the prey (hence the predator) become extinct? If so, when?

Associated to every (m×n) matrix A is the <u>rank</u> (Section III B), a natural number, the <u>null space</u> (Section III C), a set of n-vectors, the <u>range space</u> (Section III C), a set of m-vectors. Associated to some (but not all) (n×n) matrices is the <u>inverse</u> (Section III D), another (n×n) matrix.

SECTION III A: MORE MATRIX MATTERS

DEFINITION 3.1

We defined a **square matrix** in Definition 1.28: the number of rows equals the number of columns.

The **diagonal** of a square matrix (a_{ij}) is $a_{11}, a_{22}, a_{33}, \ldots$

Example 3.2 The diagonal of

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

is 1, 5, 9.

DEFINITION 3.3

A **diagonal matrix**

is a square matrix whose entries off the diagonal are zero.

Examples 3.4.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$ is a diagonal matrix;

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$ is not.

We will talk about a generalization of diagonal ("diagonalizable") matrices in Chapter VIII and use our analysis of them to solve Difference Equations.

DEFINITIONS 3.5

The **identity matrix**, denoted $\mathbf{I}$, is the diagonal matrix whose diagonal entries are all 1.

For any natural number n,

I_n denotes the $(n \times n)$ identity

matrix: $I_1 \equiv [1]$,

$$I_2 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

etc.

PROPOSITION 3.6

$IB = B$ and $AI = A$,

for any matrices A, B for which the multiplication is defined.

DEFINITION 3.7

We will often find it convenient to interchange rows and columns of a matrix.

If A is an $(m \times n)$ matrix, then the **transpose** of A, denoted A^T, is the $(n \times m)$ matrix whose i^{th} column is the i^{th} row of A, for $1 \leq i \leq m$.

Example 3.8

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 0 & -1 \end{bmatrix}.$$

3.9 PROPERTIES OF TRANSPOSE.

Suppose A and B are $(m \times n)$ matrices, C is an $(n \times k)$ matrix, for some natural numbers m, n, k, and α is a real number.

(1) $(A+B)^T = A^T + B^T$

(2) $(\alpha A)^T = \alpha A^T$

(3) $(AC)^T = C^T A^T$

(4) $(A^T)^T = A.$

DEFINITION 3.10

A matrix A is **symmetric** if $A = A^T$.

Examples 3.11

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 7 \\ 5 & 7 & \sqrt{2} \end{bmatrix}$$ is symmetric;

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$ is not.

Note that a diagonal matrix is symmetric.

See Proposition 6.48 and Remark 8.36 for clues about our interest in symmetric matrices.

REMARKS 3.12

We have mentioned in Remark 1.16 and Examples 1.23 that matrices might not commute; that is, in general, $AB \neq BA$.

An important exception is diagonal matrices.

PROPOSITION 3.13

If A and B are diagonal $(n \times n)$ matrices, then

$$AB = BA$$

Explicitly, denoting by

$$\begin{bmatrix} \alpha_1 & & & O \\ & \alpha_2 & & \\ & & \ddots & \\ O & & & \alpha_n \end{bmatrix}$$

the $(n \times n)$ diagonal matrix with diagonal entries $\alpha_1, \alpha_2, \ldots, \alpha_n$,

$$\begin{bmatrix} \lambda_1 & & & O \\ & \lambda_2 & & \\ & & \ddots & \\ O & & & \lambda_n \end{bmatrix}\begin{bmatrix} \beta_1 & & & O \\ & \beta_2 & & \\ & & \ddots & \\ O & & & \beta_n \end{bmatrix} = \begin{bmatrix} \lambda_1\beta_1 & & & O \\ & \lambda_2\beta_2 & & \\ & & \ddots & \\ O & & & \lambda_n\beta_n \end{bmatrix}$$

$$= \begin{bmatrix} \beta_1 & & & O \\ & \beta_2 & & \\ & & \ddots & \\ O & & & \beta_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & & O \\ & \lambda_2 & & \\ & & \ddots & \\ O & & & \lambda_n \end{bmatrix}$$

For other desirable properties of diagonal matrices, including their relation to Difference Equations, see the beginning of Chapter VIII.

Here's a novel connection with commuting.

PROPOSITION 3.14

If A and B are symmetric $(n \times n)$ matrices, then

(AB) is symmetric

if and only if

$AB = BA$.

Proof: If $AB = BA$, then $(AB)^T = B^T A^T = BA = AB$; that is, (AB) is symmetric.

If (AB) is symmetric, then $(AB) = (AB)^T = B^T A^T = (BA)$.

Example 3.15. Commuting is, in general, a strong requirement. To illustrate this, let's find all (2×2) matrices that commute with $A \equiv \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

SOLUTION: Denote $B \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, for a, b, c, d real (to be determined) numbers.

$$BA = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}. \text{ and}$$

$$AB = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix};$$

setting $BA = AB \longrightarrow$

$0 = c$, $a = d$, $0 = 0$, and $c = 0$,

thus

ANSWER is $\left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \middle| \begin{array}{l} a, b \text{ are} \\ \text{real} \end{array} \right\}$

SECTION III B: RANK

When discussing the set of solutions of the linear system (2.5), it is worthwhile to know that we can often focus on a special case.

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DEFINITIONS 3.16

The linear system (2.5) is

homogeneous if

$0 = b_1 = b_2 = \cdots = b_m$.

The **matrix form**

of a homogeneous linear system is

$$(3.17) \quad A\vec{x} = \vec{0}$$

where A is as in (2.15) and

$\vec{0}$ is the **zero** or

trivial vector,

the vector whose components are all zero.

Notice that a homogeneous linear system is always consistent since it always has the **trivial solution**

$0 = x_1 = x_2 = \dots$

If we have one solution of (2.5), we can write down <u>all</u> solutions by adding on solutions of the corresponding homogeneous linear system.

<u>THEOREM 3.18</u>

If $\vec{x}_0$ is a solution of $A\vec{x} = \vec{b}$, then the set of <u>all</u> solutions of $A\vec{x} = \vec{b}$ is

$$\left\{ (\vec{x}_0 + \vec{x}) \mid A\vec{x} = \vec{0} \right\}$$

Proof: If $A\vec{x} = \vec{0}$, then

$$A(\vec{x}_0 + \vec{x}) = A\vec{x}_0 + A\vec{x} = \vec{b} + \vec{0}$$
$$= \vec{b}.$$

Conversely, if $A\vec{y} = \vec{b}$, then
$A(\vec{y} - \vec{x}_0) = A\vec{y} - A\vec{x}_0 = \vec{b} - \vec{b}$
$= \vec{0}$, thus
$\vec{y} = (\vec{x}_0 + (\vec{y} - \vec{x}_0))$, with $A(\vec{y} - \vec{x}_0) = \vec{0}$.

REMARK 3.19. Theorem

3.18 implies that, when counting or describing the number of solutions of a linear system, we may assume it is homogeneous.

The following result, sort of answering the question "how many solutions could a linear system have," was illustrated in Remark 2.10.

THEOREM 3.20

The number of solutions a linear system could have is 0, 1, or ∞.

Proof: An inconsistent linear system, such as "$1 = 0$" has 0 solutions.

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The linear system "$x_1 = 0$" has precisely one solution.

What remains is to show that, if a linear system has two solutions, then it has infinitely many solutions. So suppose (using the matrix form (2.15)) that $A\vec{x}_1 = \vec{b} = A\vec{x}_2$, with $\vec{x}_1 \neq \vec{x}_2$. Then, for any real s,

$$A(s\vec{x}_1 + (1-s)\vec{x}_2) =$$

$$s A\vec{x}_1 + (1-s) A\vec{x}_2 =$$

$$s\vec{b} + (1-s)\vec{b} = \vec{b};$$

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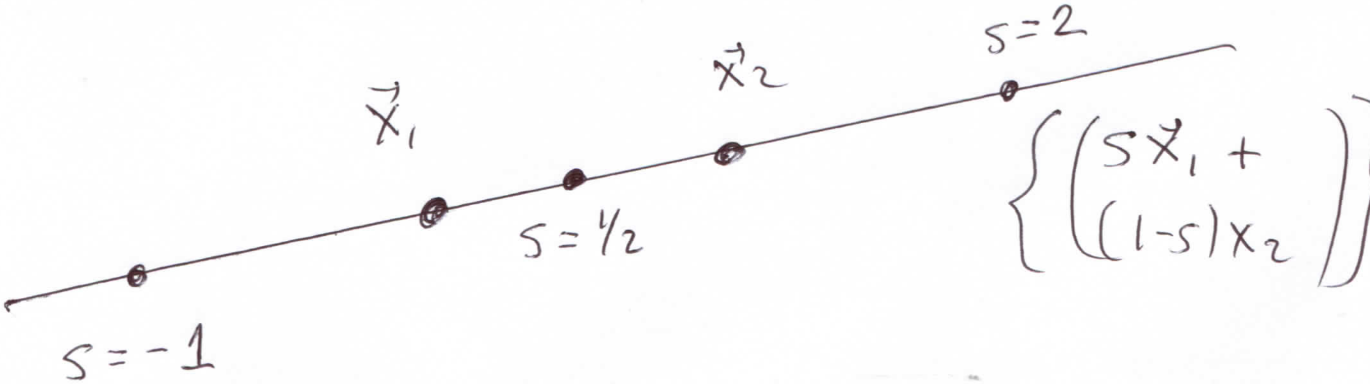
$(s\vec{x}_1 + (1-s)\vec{x}_2)$ is a solution of $A\vec{x} = \vec{b}$.

This produces infinitely many solutions of $A\vec{x} = \vec{b}$



HOW MANY SOLUTIONS of a linear system?

0	1	∞
inconsistent	unique solution	nonunique solutions
	0 free variables	>0 free variables

HOW MANY SOLUTIONS of a homogeneous linear system?

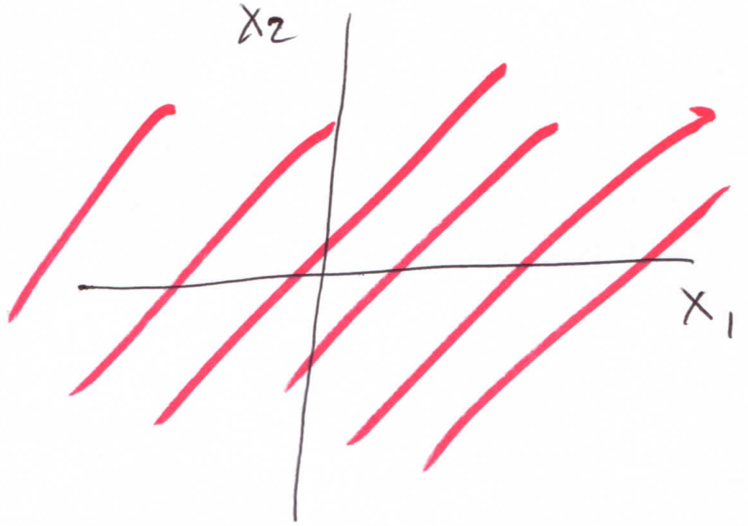
1	∞
only trivial soln.	some nontrivial solns.
0 free variables	> 0 free variables

REMARKS 3.21

We should not be satisfied with the "∞" part of Theorem 3.20, because there are different infinities.

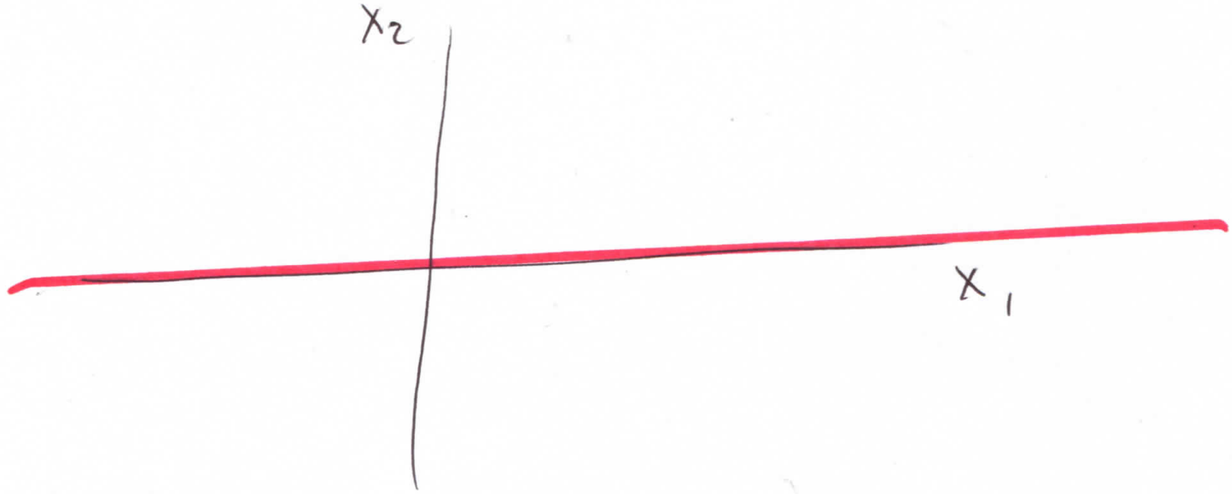
Consider the ∞ of the plane

$$\mathbb{R}^2 \equiv \{(x_1, x_2) \mid x_1, x_2 \text{ are real}\}$$



compared to the ∞ of the line

$$\{(x_1, 0) \mid x_1 \text{ is real}\}$$



The line seems to have a smaller ∞ than the plane, as expressed in the decreased freedom when forced to move only along the line.

We can best describe the difference between the plane and the line with the idea of <u>free variables</u> from Definition 2.13: the plane above has two free variables x_1 and x_2, while the line has only one, x_1.

GOAL 3.22

Without solving, count the number of free variables in {solutions} of a consistent linear system.

INTUITION 3.23

(not necessarily correct)

$$\left(\text{number of free variables}\right) = \left(\begin{array}{c}\text{number}\\ \text{of variables}\end{array}\right) - \left(\begin{array}{c}\text{number}\\ \text{of equations}\end{array}\right)$$

Examples 3.24

$$\text{Space} \equiv \mathbb{R}^3 \equiv \left\{ (x_1, x_2, x_3) \,\middle|\, \begin{array}{c} x_1, x_2, \\ x_3 \text{ real} \end{array} \right\}$$

has 3 free variables.

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The linear system

$$x_3 = 0$$

(Authority figure mandates "NO FLOATING")

has 2

$$\left(\underset{\substack{\uparrow \\ \text{number of variables}}}{3} - \underset{\substack{\uparrow \\ \text{number of equations}}}{1}\right) = 2 \text{ free variables}$$

x_1, x_2

The linear system

$$\begin{aligned} x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

has

$$\left(\underset{\substack{\uparrow \\ \text{number of variables}}}{3} - \underset{\substack{\uparrow \\ \text{number of equations}}}{2}\right) = 1 \text{ free variable } x_1$$

Finally and most restrictively, the linear system

$$\begin{aligned} x_1 \quad\quad\quad &= 0 \\ x_2 \quad\quad &= 0 \\ x_3 &= 0 \end{aligned}$$

has $(\underset{\substack{\text{number}\\\text{of}\\\text{variables}}}{3} - \underset{\substack{\text{number}\\\text{of}\\\text{equations}}}{3}) = 0$ free variables (unique solution $\vec{x} = \vec{0}$)

Each equation should be thought of as a rope around your neck, restricting your movement and the variability of the variables; each equation is an additional loss of freedom or free variables.

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THEOREM 3.25

Intuition 3.23 is **false**, in general.

Proof: The linear system

$$x_1 + x_2 = 1$$
$$2x_1 + 2x_2 = 2$$

has 2 equations and 2 variables, but $\{\text{solutions}\}$ $= \{(1 - x_2, x_2) \mid x_2 \text{ is real}\}$ has 1 free variable, and

$$(\underset{\substack{\uparrow \\ \text{number of} \\ \text{variables}}}{2} - \underset{\substack{\uparrow \\ \text{number} \\ \text{of equations}}}{2}) \neq \underset{\substack{\nwarrow \\ \text{number of} \\ \text{free variables}}}{1}$$

PROBLEM:

The linear system above is "really" only one equation; the second equation is redundant.

The redundancy might be much more effectively buried; recall Examples 2.30(d), where

$$x_1 + x_2 - x_3 = 2$$

$$x_1 - x_2 = 1$$

$$2x_1 - x_3 = 3$$

turned out to be equivalent

to

$$
\begin{aligned}
x_1 \qquad - \tfrac{1}{2} x_3 &= 3/2 \\
x_2 - \tfrac{1}{2} x_3 &= 1/2 \\
0 &= 0
\end{aligned}
$$

"really" only two equations, although it began as three equations.

Our {solutions} turned out to have $(3 - 2) = 1$ free variable,

number of variables — "real" number of equations

not $(3 - 3) = 0$ free variables

number of variables — number of equations

To answer the question of

HOW MANY equations "really,"

we need the following definition.

DEFINITION 3.26

The **rank** of a matrix A,

denoted **rank(A)**, sometimes

abbreviated as **r(A)** or **r**

is calculated as follows.

1. Use elementary operations to put A in echelon form.

2. rank(A) is then the number of nontrivial (not all zeroes) rows.

Examples 3.27

(i) $\operatorname{rank}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \overset{(R_2 - 2R_1)}{=}$

$\operatorname{rank}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 1.$

This explains why the linear system in the proof of Theorem 3.25 has 1 free variable in its {solutions}:

$$\underset{\substack{\text{number}\\ \text{of variables}}}{(2} - \underset{\text{rank}}{1)} = \underset{\substack{\text{number of}\\ \text{free variables}}}{1}$$

(2) $\text{rank}\begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & -1 & 0 & 1 \\ 2 & 0 & -1 & 3 \end{bmatrix} = \begin{pmatrix} R_2 - R_1 \\ R_3 - 2R_1 \end{pmatrix}$

$= \text{rank}\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & 1 & -1 \end{bmatrix} = (R_3 - R_2)$ (then $-\frac{1}{2}R_2$)

$\text{rank}\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 2$;

this explains why the linear system in the "Problem" after the proof of Theorem 3.25 has

$$\underset{\substack{\uparrow \\ \text{number of variables}}}{3} - \underset{\substack{\uparrow \\ \text{rank}}}{2} = \underset{\substack{\uparrow \\ \text{number of free variables}}}{1}$$

in its {solutions}.

We will now use rank to correct our false Intuition 3.23. As a bonus, we will characterize consistency in terms of rank.

CORRECTED INTUITION 3.28

(a) A linear system is consistent if and only if

$$\text{rank}\begin{pmatrix}\text{coeffi-}\\\text{cient}\\\text{matrix}\end{pmatrix} = \text{rank}\begin{pmatrix}\text{aug-}\\\text{mented}\\\text{matrix}\end{pmatrix}$$

(b) If consistent, then

$$\begin{pmatrix}\text{number of free variables}\end{pmatrix} = \begin{pmatrix}\text{number of}\\\text{variables}\end{pmatrix} - \text{rank}\begin{pmatrix}\text{coefficient}\\\text{matrix}\end{pmatrix}$$

$$\equiv \quad n \quad - \quad \underset{\begin{pmatrix}\sim\text{ “real” number}\\\text{of equations}\end{pmatrix}}{r}$$

Examples 3.29. In each of the following, determine if the linear system is consistent; if it is, find the number of free variables in {solutions}.

(a)
$$
\begin{aligned}
x_1 \qquad\quad + x_3 - x_4 \qquad\quad &= 0 \\
x_1 - x_2 \qquad\quad + x_4 - x_5 &= 1 \\
2x_1 - x_2 + x_3 \qquad\quad - x_5 &= 1 \\
x_2 + x_3 - 2x_4 + x_5 &= 0
\end{aligned}
$$

SOLUTION: We can calculate ranks of coefficient and augmented matrices simultaneously:

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$$\text{rank}\left[\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 \\ 2 & -1 & 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -2 & 1 & 0 \end{array}\right] =$$

$$\begin{pmatrix} R_2 - R_1 \\ R_3 - 2R_1 \end{pmatrix} \quad \text{rank}\left[\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 1 \\ 0 & -1 & -1 & 2 & -1 & 1 \\ 0 & 1 & 1 & -2 & 1 & 0 \end{array}\right] =$$

$$\begin{pmatrix} R_3 - R_2 \\ R_4 + R_2 \end{pmatrix} \quad \text{rank}\left[\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] =$$

$$\begin{pmatrix} -R_2 \\ R_3 \leftrightarrow R_4 \end{pmatrix} \quad \text{rank}\left[\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow$$

$$\text{rank}\begin{pmatrix} \text{coefficient} \\ \text{matrix} \end{pmatrix} = 2 \neq 3 = \text{rank}\begin{pmatrix} \text{augmented} \\ \text{matrix} \end{pmatrix}$$

$\longrightarrow$ $\boxed{\text{inconsistent}}$

(b)
$$\begin{aligned}
x_1 \quad\quad\;\; + x_3 - x_4 \quad\quad\;\; &= 0 \\
x_1 - x_2 \quad\quad\;\; + x_4 - x_5 &= 1 \\
2x_1 - x_2 + x_3 \quad\quad\;\; - x_5 &= 1 \\
x_2 + x_3 - 2x_4 + x_5 &= -1
\end{aligned}$$

SOLUTION: As with (a),

$$\operatorname{rank}\left[\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 \\ 2 & -1 & 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -2 & 1 & -1 \end{array}\right] = \cdots$$

$$\operatorname{rank}\left[\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right];$$

both the coefficient matrix and the augmented matrix have rank $r = 2$, so the system is $\boxed{\text{consistent}}$, with

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number of free variables $= (n-r)$

$= (5-2) = 3$

number of variables

$r \sim$ "real" number of equations

Please notice that (a) differs from (b) only in the lower right number (-1) being replaced by 0.

Consistency can be disturbingly unstable; if (-1) in (b) were replaced by any other number, the linear system would be inconsistent.

(c) $A\vec{x} = \vec{b}$, where

A is a (10×19) matrix,

$\text{rank}(A) = 7$ and $\text{rank}[A \ \vec{b}] = 8$.

SOLUTION: inconsistent since the rank of the coefficient matrix is 7, different from 8, the rank of the augmented matrix.

(d) SAME as (c), except $\text{rank}[A \ \vec{b}] = 7$.

Now that coeff. and augmented matrices have same rank, system is consistent.

$$\begin{pmatrix}\text{number of columns of} \\ \text{coefficient matrix}\end{pmatrix}$$

$$= \begin{pmatrix}\text{number of variables in} \\ \text{linear system}\end{pmatrix},$$

so

$$\boxed{\begin{pmatrix}\text{number of free variables}\end{pmatrix} = (19 - 7) = 12}$$

19: number of variables $\equiv n$

7: "real" number of equations $\equiv r$

Notice that 10 = number of equations was not relevant.

RANK PROPERTIES

3.30

If A is an $(m \times n)$ matrix, then

(1) $\text{rank}(A) = \text{rank}(A^T)$

(2) $\text{rank}(A) \leq m = \begin{pmatrix} \text{number of} \\ \text{rows of } A \end{pmatrix}$

(3) $\text{rank}(A) \leq n = \begin{pmatrix} \text{number of} \\ \text{columns of } A \end{pmatrix}$

If A is the coefficient matrix of a linear system, then

m = number of equations

n = number of variables

Examples 3.31

In each of the following, say as much as possible about {solutions} just from the information given.

(1) A system of 7 equations, with 10 variables.

(2) A homogeneous system of 7 equations with 10 variables.

(3) A system of 10 equations with 7 variables.

(4) A homogeneous system of 10 equations with 7 variables.

SOLUTIONS. In each part, let $A \equiv$ coefficient matrix, and, as in 3.28,

$n \equiv$ number of columns of A

$=$ number of variables

$r \equiv$ rank of A

(i) COULD BE NO SOLUTIONS,

e.g.,

$$\begin{aligned} 0 &= 1 \\ 0 &= 0 \\ &\vdots \\ 0 &= 0 \end{aligned}$$

If consistent, then 3.28

implies

$$(10 - r)$$

free variables, where $r \leq 7$ and $r \leq 10$

$\rightarrow \quad 0 \leq r \leq 7$

$\rightarrow \quad (10-7) \leq (10-r) \leq (10-0)$;

thus

$$\boxed{3 \leq \begin{pmatrix}\text{number of} \\ \text{free variables}\end{pmatrix} \leq 10.}$$

In particular, there could be infinitely many solutions since the number of free variables could be > 0.

There cannot be a unique solution

since the number of free variables cannot equal 0; so we can make a stronger statement: If consistent, must be infinitely many solutions

(2) homogeneous $\rightarrow$

must be at least 1 solution.

namely the trivial solution

$0 = x_1 = x_2 = \ldots = x_{10}$.

Arguing as in (1), $3 \leq$ number of free vars ≤ 10

there must be infinitely many solutions;

in the setting of homogeneous linear systems, this is equivalent to there are nontrivial solutions,

(3) As in (1), could be no solution;

otherwise, $0 \leq r \leq 7$,

but now $n \equiv$ number of variables $= 7$, so 3.28 implies that

$0 = (7-7) \leq$ number of free variables $\leq 7 - 0 = 7$

In particular

there could be a unique solution

since there could be 0 free variables; and

there could be infinitely many solutions

since there could be > 0 free variables

(4) homogeneous $\rightarrow$ (as in (2))

must be at least one solution

As in (3)

$0 \leq$ number of free variables ≤ 7

and there could be a unique solution or only the trivial solution is possible; also there could be infinitely many solutions or could be nontrivial solutions.

REMARKS 3.32

The solution of a linear system is reality, possibly hidden before we applied our techniques. Thus it is disturbing when a linear system is inconsistent, that is, has no solution; absence of reality seems too postmodern. When a linear system is inconsistent, we assume mistakes occurred when constructing the linear model.

Also not ideal, unless we're prepared to consider parallel universes, is to have more than one solution, meaning more than one reality.

Existence of a <u>unique</u> solution of a linear system is our favorite outcome.

Below we've used 3.28 to characterize having a unique solution, along with some other scenarios of interest.

3.33 SOME COROLLARIES OF 3.28.

(1) A consistent linear system has a unique solution
if and only if
it has 0 free variables
if and only if
the rank of the coefficient matrix equals the number of variables.

(2) If the number of variables in a consistent linear system is greater than the number of equations, then the solutions are not unique.

(3) If the number of variables in a homogeneous linear system is greater than the number of equations, then the system has nontrivial solutions.

Proof: In each part, denote by A the coefficient matrix, by n the number of columns of A = the number of variables, by m the number of rows of A = the number of equations, by r the rank of A.

(1) is clear from 3.28:

$(n-r)$ = number of free variables.

(2) and (3) follow from 3.28 and 3.30: $r \le m$, so that

$n > m \rightarrow n > r \rightarrow$

more than 0 free variables.

REMARKS 3.34

Very informally, rank measures the amount of information in a matrix. When the matrix is the coefficient matrix of a linear system, this information is negative in the sense of restricting the possible solutions (see 3.28 (b)).

The characterization of

$$A\vec{x} = \vec{b}$$

being consistent when

$$\mathrm{rank}(A) = \mathrm{rank}([A \ \vec{b}])$$

may also be thought of in terms of information: the vector $\vec{b}$ is not contributing any new information.

In Difference Equations, rank will play a more positive role, quantifying how much information is passed on from generation to generation. See in particular Section IV.E, where we introduce dimension and calculate the dimension of an object to be introduced in Section III C, the range space.