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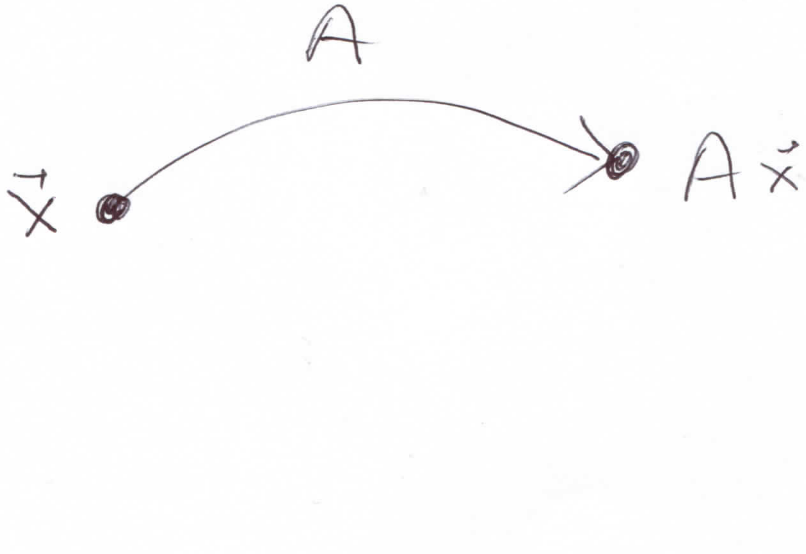
CHAPTER V: DETER-MINANTS

DISCUSSION and DEFINITIONS 5.1

Throughout this chapter, A is an $(n \times n)$ matrix, for n a natural number.

n is then the **order** of A.

We have seen A acting on individual vectors:



We also have a name for the vector space formed by letting A act on all vectors in $\mathbb{R}^n$, the <u>range space</u> of A

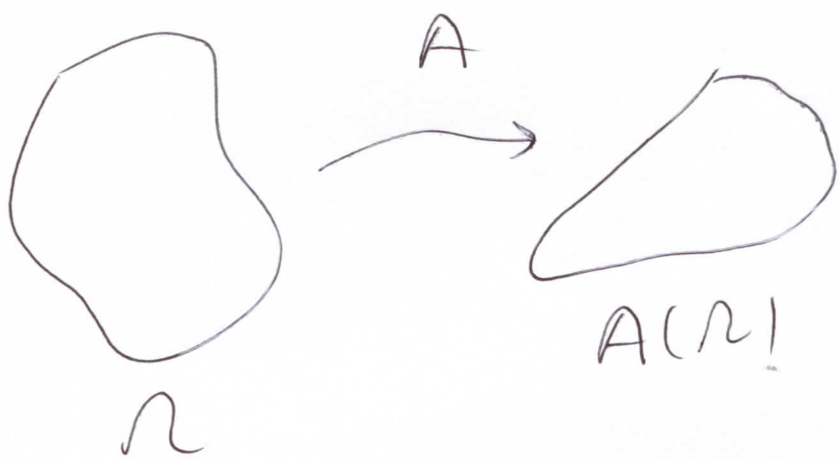
$$R(A) = \{ A\vec{x} \mid \vec{x} \text{ is in } \mathbb{R}^n \}$$

(see (3.42) and (3.43)).

We'd like to take an intermediate approach. For any Ω contained in $\mathbb{R}^n$, define

$A(\Omega)$ (reads "A of Ω")

$\equiv \{ A\vec{x} \mid \vec{x} \text{ is in } \Omega \}$.



Of particular interest is Ω chosen to be

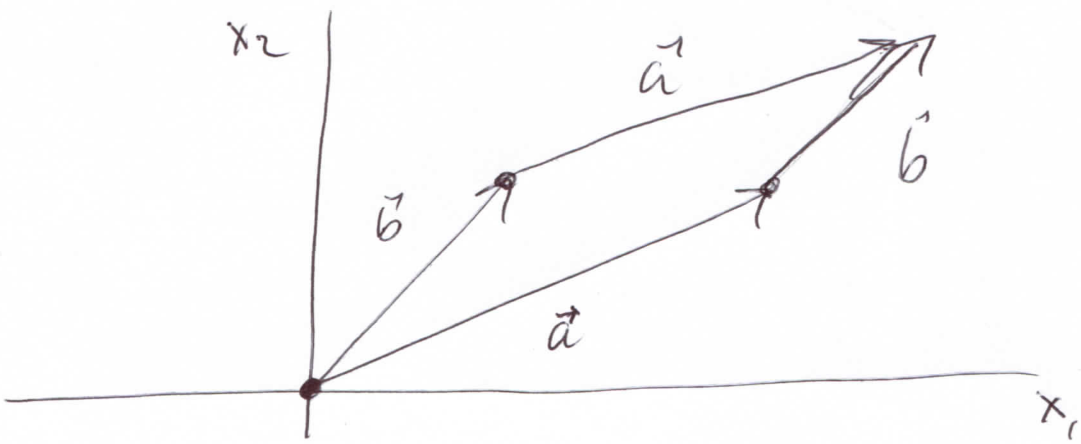
$$C_n \equiv \left\{ \vec{x} = (x_1, x_2, \ldots, x_n) \;\middle|\; \begin{array}{c} 0 \le x_j \le 1 \\ (1 \le j \le n) \end{array} \right\}$$

The letter C stands for "cube," as when $n=3$.

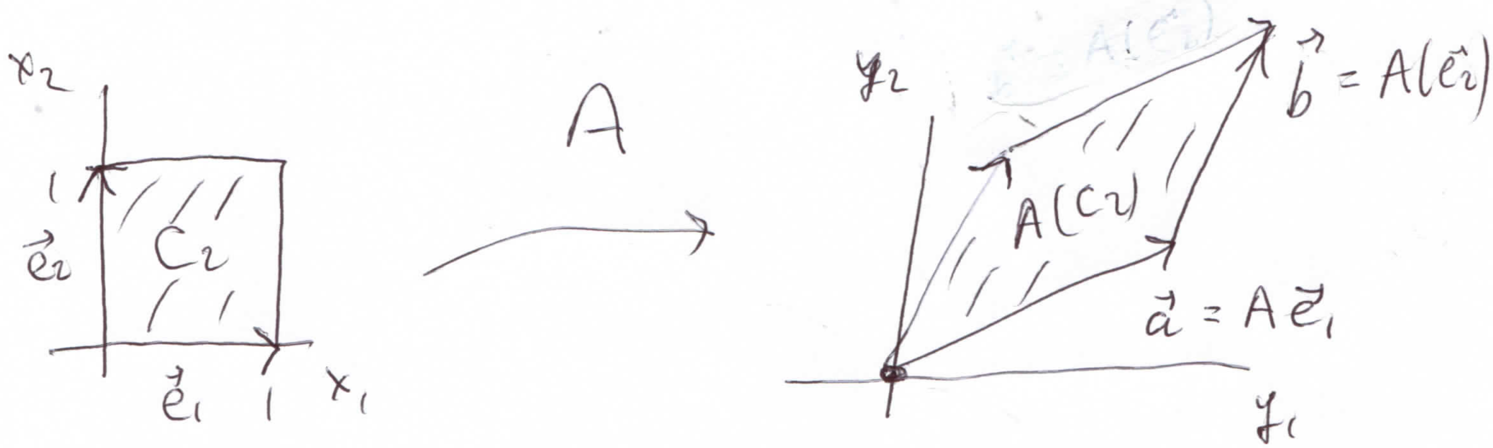
The remainder of this discussion will be for $n=2$, where we can draw pictures, but it is not hard to generalize to arbitrary n.

For $\vec{a}$ and $\vec{b}$ vectors in $\mathbb{R}^2$, the **parallelogram formed by $\vec{a}$ and $\vec{b}$**

is the quadrilateral with vertices $\vec{0}$, $\vec{a}$, $\vec{b}$, and $(\vec{a} + \vec{b})$



It can be shown that, if $A = [\vec{a} \ \vec{b}]$, then $A(C_2)$ is the parallelogram formed by $\vec{a}$ and $\vec{b}$.



The <u>area</u> of $A(C_2)$ is useful information about A; it can be shown that, for (almost) any Ω contained in $\mathbb{R}^2$,

(area of $A(\Omega)$) =

(area of Ω)(area of $A(C_2)$)

For this reason, $A(C_2)$ is called a

magnification factor

for A.

In Drawings 5.2 (a)-(d) we've drawn $A(C_2)$ for different choices of A.

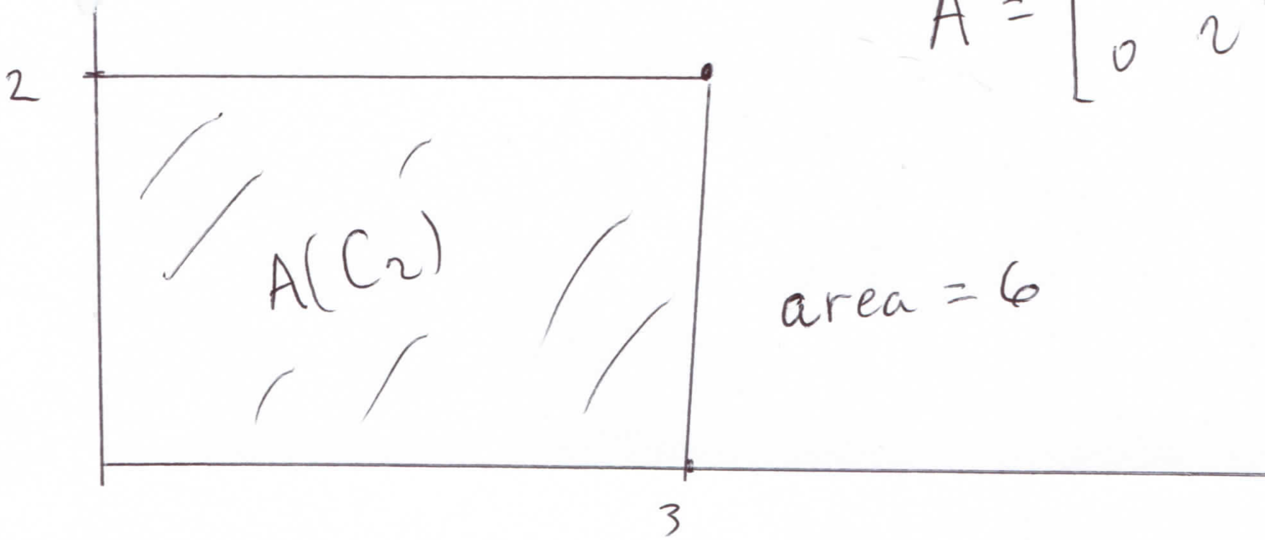
In Drawings 5.(2)(a), where $A(C_2)$ is a rectangle, we could get the area by multiplying the diagonal entries:

$$3 \cdot 2 = 6.$$

This initial guess at a way of calculating the area of $A(C_2)$ breaks down in (b)-(d), as the diagonal entries stay the same but the areas shrink as the columns of A rotate towards each other. Apparently the off-diagonal entries are diminishing the area. A precise statement of this will appear in our definition of <u>determinant</u> (Definition 5.5)

DRAWINGS 5.2

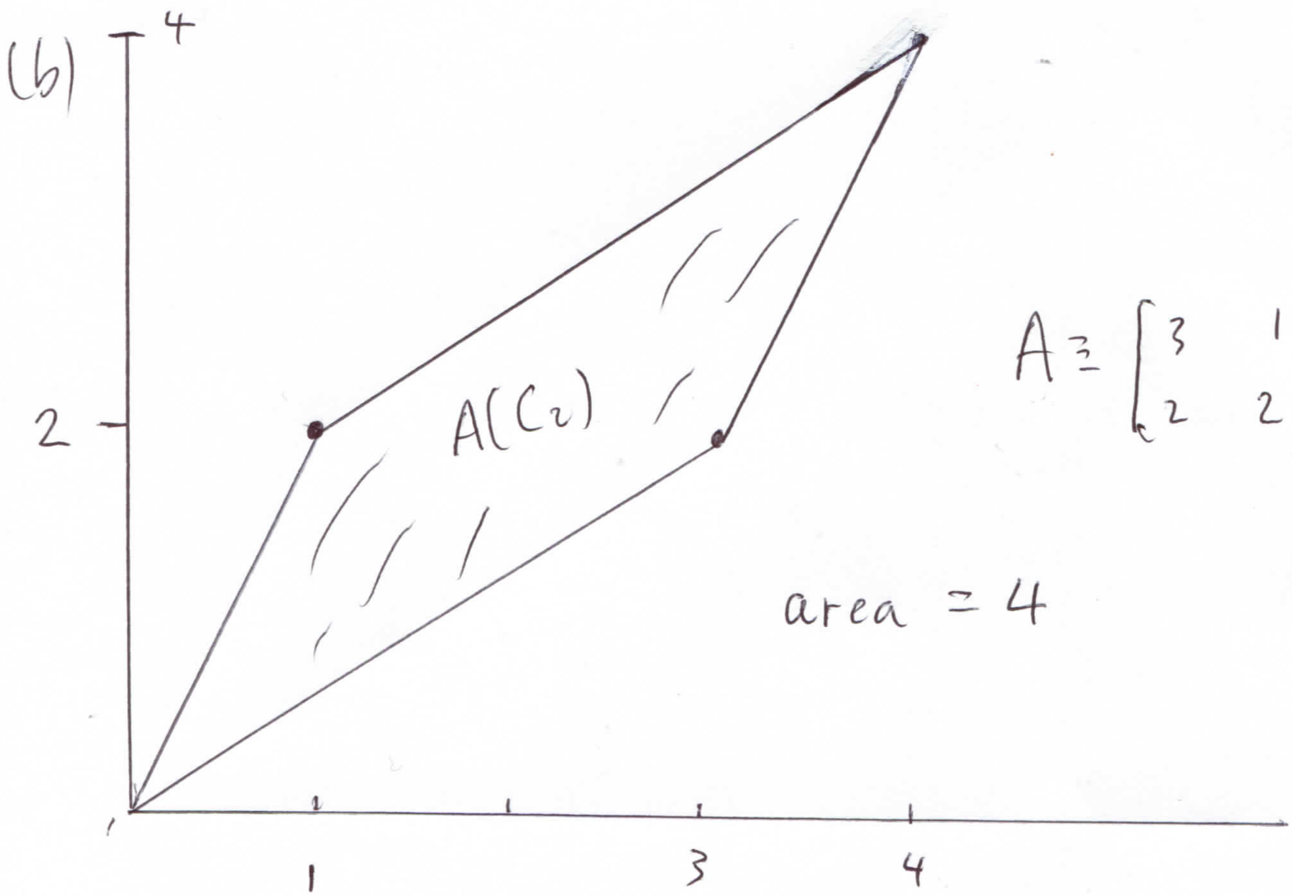
(a)



$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

area = 6

(b)

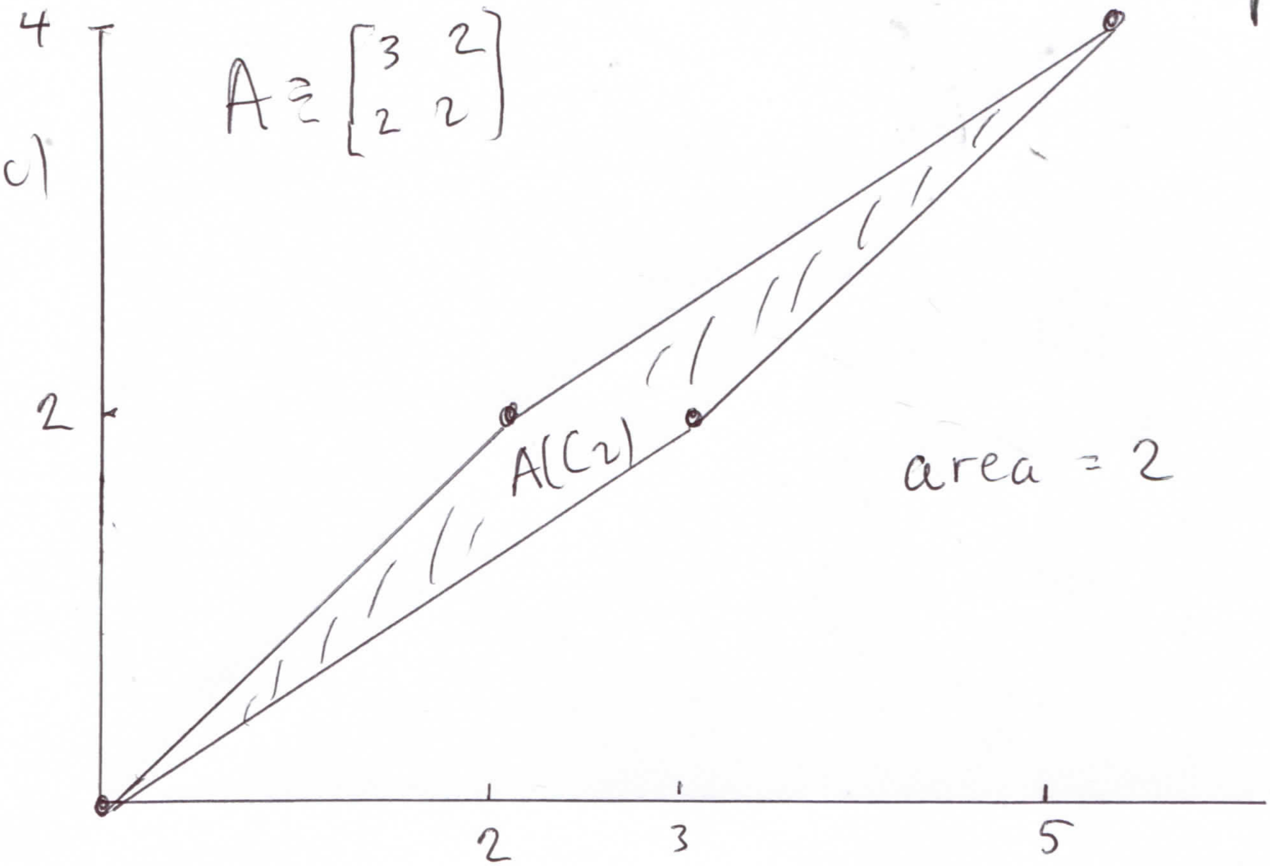


$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$

area = 4

(c)

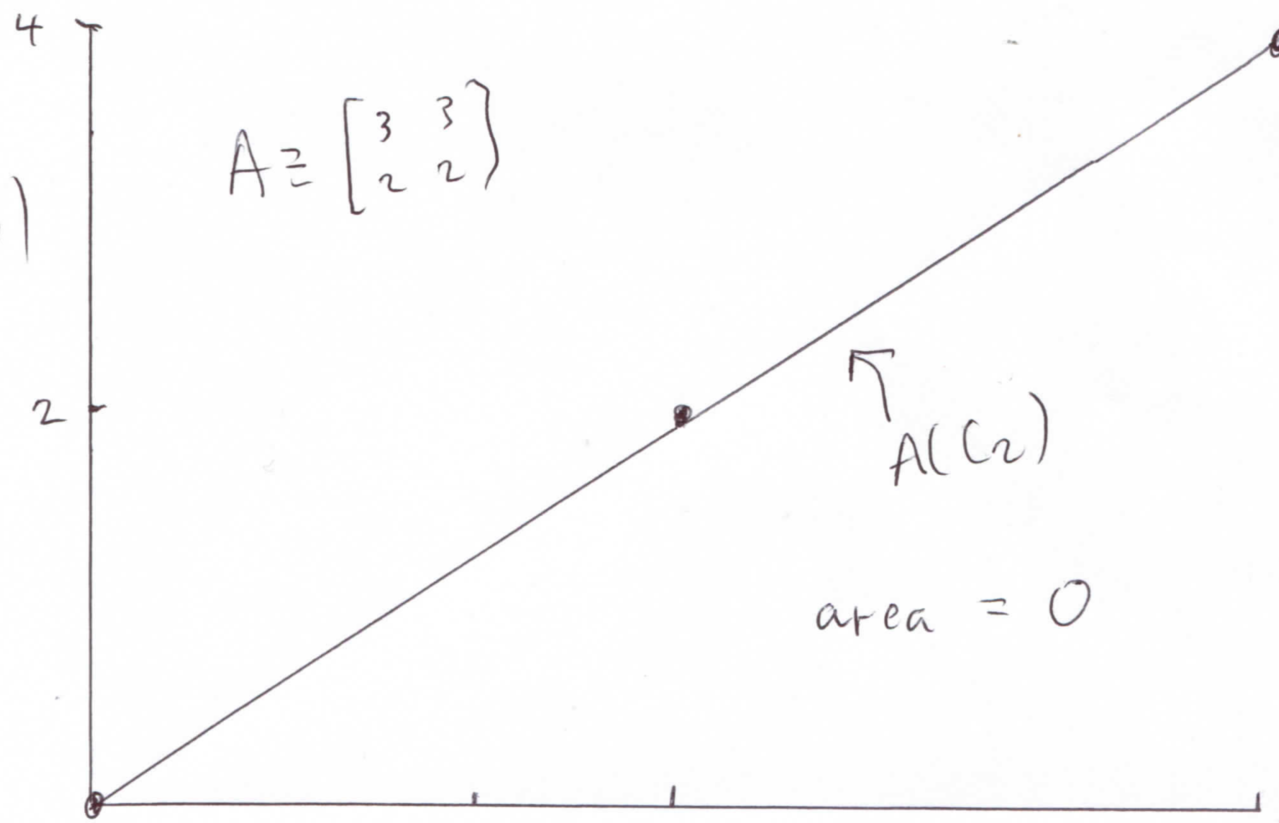
$A = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$



area = 2

(d)

$A = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$



area = 0

DEFINITIONS 5.3.

A **submatrix** of a matrix B is a matrix formed from B by removing (zero or more but not all) rows or columns of B.

In particular, if B is $(m \times n)$, $1 \le i \le m$, and $1 \le j \le n$, then

$B_{ij} \equiv B$, after removing the i^{th} row and the j^{th} column.

Examples 5.4

Let $B = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & 4 & 3 & 0 \\ 2 & 3 & 4 & 5 \end{bmatrix}$.

Then $B_{32} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \end{bmatrix}$

Some other submatrices of B

are $\begin{bmatrix} 1 & 0 & -1 \\ 4 & 3 & 0 \\ 3 & 4 & 5 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \end{bmatrix}$,

$\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$

DEFINITION 5.5

det A or **det(A)**, short for

determinant of A

is defined for any square matrix A recursively, as follows.

$$\det[a] \equiv a,$$

for any real number a.

For $n = 2, 3, \ldots$, if det of $((n-1)\times(n-1))$ matrices is defined, then for $A \equiv (a_{ij})$ an $(n \times n)$ matrix,

$$\det A \equiv \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det(A_{1j})$$

$$\equiv a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13}) - \cdots$$

As with the solution of a Difference Equation, a domino effect defines det A for any square matrix A.

Examples 5.6

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} =$$

$$a_{11} \det A_{11} - a_{12} \det A_{12}$$

$$= a_{11} a_{22} - a_{12} a_{21}.$$

In particular,

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2.$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} =$$

$$a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \leftarrow A_{11}$$

$$- a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \leftarrow A_{12}$$

$$+ a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \leftarrow A_{13}$$

$$= a_{11} (a_{22} a_{33} - a_{23} a_{32})$$

$$- a_{12} (a_{21} a_{33} - a_{23} a_{31})$$

$$+ a_{13} (a_{21} a_{32} - a_{22} a_{31})$$

$$\det\begin{bmatrix} 2 & 3 & 0 & 4 \\ 5 & 6 & 0 & 1 \\ 7 & 0 & 0 & 1 \\ 8 & 9 & 1 & 10 \end{bmatrix} \doteq$$

$$2\det\begin{bmatrix} 6 & 0 & 1 \\ 0 & 0 & 1 \\ 9 & 1 & 10 \end{bmatrix} - 3\det\begin{bmatrix} 5 & 0 & 1 \\ 7 & 0 & 1 \\ 8 & 1 & 10 \end{bmatrix}$$

$$+ 0\det\begin{bmatrix} 5 & 6 & 1 \\ 7 & 0 & 1 \\ 8 & 9 & 10 \end{bmatrix} - 4\det\begin{bmatrix} 5 & 6 & 0 \\ 7 & 0 & 0 \\ 8 & 9 & 1 \end{bmatrix} =$$

$$2\left[6\det\begin{bmatrix} 0 & 1 \\ 1 & 10 \end{bmatrix} - 0\det\begin{bmatrix} 0 & 1 \\ 9 & 10 \end{bmatrix} + 1\det\begin{bmatrix} 0 & 0 \\ 9 & 1 \end{bmatrix}\right]$$

$$-3\left(5\det\begin{bmatrix} 0 & 1 \\ 1 & 10 \end{bmatrix} + 0\det\begin{bmatrix} 7 & 1 \\ 8 & 10 \end{bmatrix} + 1\det\begin{bmatrix} 7 & 0 \\ 8 & 1 \end{bmatrix}\right) + 0$$

$$-4\left(5\det\begin{bmatrix} 0 & 0 \\ 9 & 1 \end{bmatrix} - 6\det\begin{bmatrix} 7 & 0 \\ 8 & 1 \end{bmatrix} + 0\det\begin{bmatrix} 7 & 0 \\ 8 & 9 \end{bmatrix}\right)$$

$$= 2(-6) - 3(-5+7)$$

$$-4(-42) = \boxed{150}$$

There are many ways to calculate det; here's a sample.

PROPOSITION 5.7

(1) For any j, $1 \leq j \leq n$,

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{ij})$$

(2) For any i, $1 \leq i \leq n$

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{ij})$$

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5.8 EFFECTS OF ELEMENTARY OPERATIONS ON DET.

(1) $R_i \leftrightarrow R_j$: det is multiplied by (-1)

(2) kR_i : det is multiplied by k

(3) $R_i + kR_j$ $(i \neq j)$: det is unchanged.

Examples 5.9

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 4-6 = (-2) = (-1)\,2$$

$$= (-1) \det \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

$$\det\begin{bmatrix} 4 & 8 \\ 3 & 4 \end{bmatrix} = 16 - 24 = -8$$

$$= 4(-2) = 4 \det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\det\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = (-2) = \det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \right)$$

MORE PROPERTIES OF DET 5.10

(1) $\det(A^T) = \det A$

(2) $\det(AB) = (\det A)(\det B)$

if A and B are $(n \times n)$ matrices

(3) If all entries below the diagonal of A are zero, then det A equals the product of the diagonal entries.

(4) If a row or column of A is the zero vector, then det $A = 0$.

(5) If two rows or two columns of A are the same, then det $A = 0$.

(6) (Cramer's rule) If det $A \neq 0$, then, ~~for~~ any n-vector $\vec{b}$, $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = (x_1, x_2, \ldots, x_n)$, where,

for $1 \le j \le n$,

$$x_j = \frac{\det A_j}{\det A},$$

where A_j is A, after replacing the j^{th} column of A with $\vec{b}$.

(7) If $A = [\vec{a}\ \vec{b}]$, for column 2-vectors $\vec{a}$ and $\vec{b}$, then the parallelogram formed by $\vec{a}$ and $\vec{b}$ has area $|\det A|$.

REMARKS 5.11. The primary interest of Cramer's rule is that det A nonzero is guaranteeing a solution of $A\vec{x} = \vec{b}$.

Property (3) of 5.10, combined with 5.8, implies a strategy for calculating $\det A$: use elementary operations to put A in echelon form. In Chapter VIII we will prefer calculating det using our definition.

Notice that 5.8 makes Property (5) a consequence of Property (4).

Compare Property (7) of 5.10 with the areas asserted in Drawings 5.2.

Chapter VIII will rely on characterization of invertibility of a matrix. Below we enhance parts of 4.53 by adding on a determinant characterization of invertibility.

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THEOREM 5.12

The following are equivalent; that is, if one assertion is true, then all the others are true. A is an $(n \times n)$ matrix.

(1) A is nonsingular.

(2) A is invertible.

(3) $A\vec{x} = \vec{b}$ has a solution, for any n-vector $\vec{b}$.

(4) rank $A = n$.

(5) det $A \neq 0$.

Then $A\vec{x} = \vec{b}$ has the unique solution

$$\vec{x} = A^{-1}\vec{b}.$$

~ **Proof:** We'll use "$\leftrightarrow$" for "if and only if".

(1) $\leftrightarrow$ (3). Recall (Corollary 4.50) that $\dim(\mathcal{N}(A)) = n - \dim(\mathcal{R}(A))$. Since (1) is equivalent to $\dim(\mathcal{N}(A)) = 0$ and (3) is equivalent to $\mathcal{R}(A) = \mathbb{R}^n$, the equivalence follows.

(3) $\leftrightarrow$ (4) $\operatorname{rank} A = \dim(R(A))$, which equals $n \leftrightarrow R(A) = \mathbb{R}^n$, which, as we're already mentioned, is equivalent to (3).

(2) $\to$ (5) $\underline{1} = \det(I_n) = \det(AA^{-1}) = (\det A)(\det(A^{-1}))$ $\to \det A \neq 0$.

(5) $\to$ (3) Cramer's Rule.

(3) $\to$ (2) The inverse matrix may be constructed: for $1 \leq j \leq n$, let $\vec{v}_j$ be the solution of

$$A\vec{x} = \vec{e}_j;$$

then it may be shown that

$$[\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n]$$

is the inverse of A.

It might seem surprising that rank may be characterized in terms of det.

PROPOSITION 5.13

If A is a (not necessarily square) matrix, then rank A is the largest order of square submatrices of A with nonzero det.

Example 5.14

Let $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 0 & 2 \end{bmatrix}$

All (3×3) submatrices have zero det, but the submatrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has nonzero det and is (2×2). Thus rank $A = 2$.

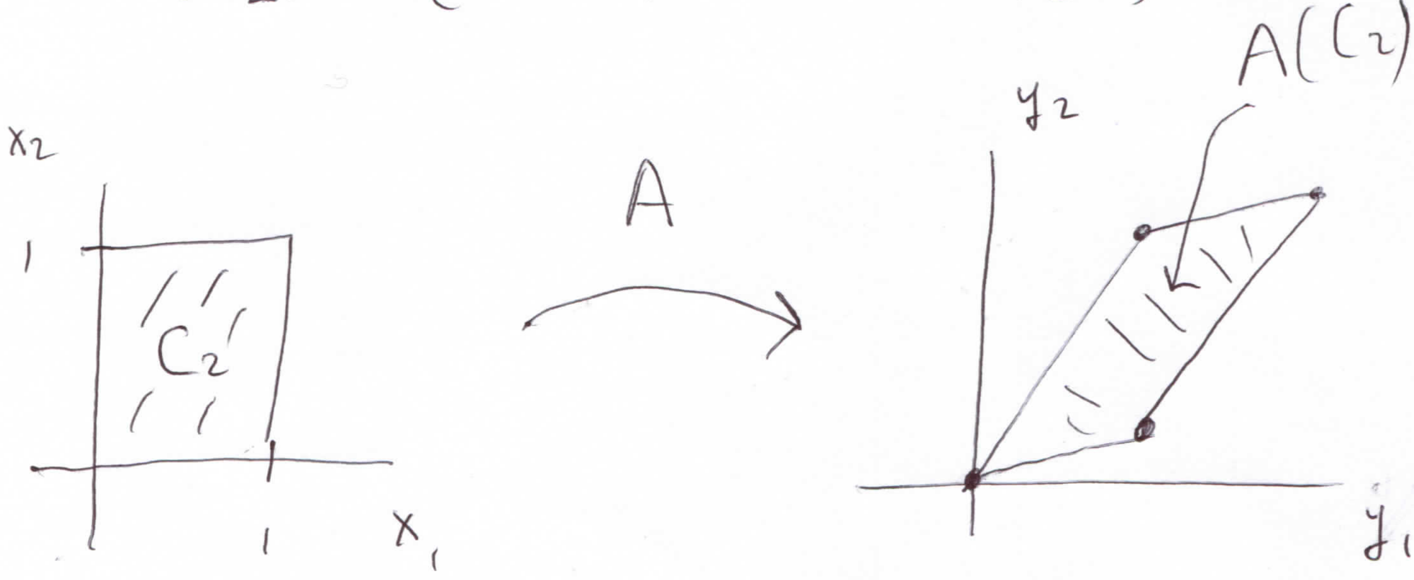
REMARKS 5.15

The area statement of 5.10(7) gives a clue to the intuition of Theorem 5.12 (2) $\leftrightarrow$ (5), at least when $n=2$.

We talked in 5.1 about

$C_2 \equiv \{ (x_1, x_2) \mid 0 \le x_j \le 1,\ j = 1, 2 \}$ and

$A(C_2) \equiv \{ A\vec{x} \mid \vec{x}$ is in $C_2 \}$



area = 1

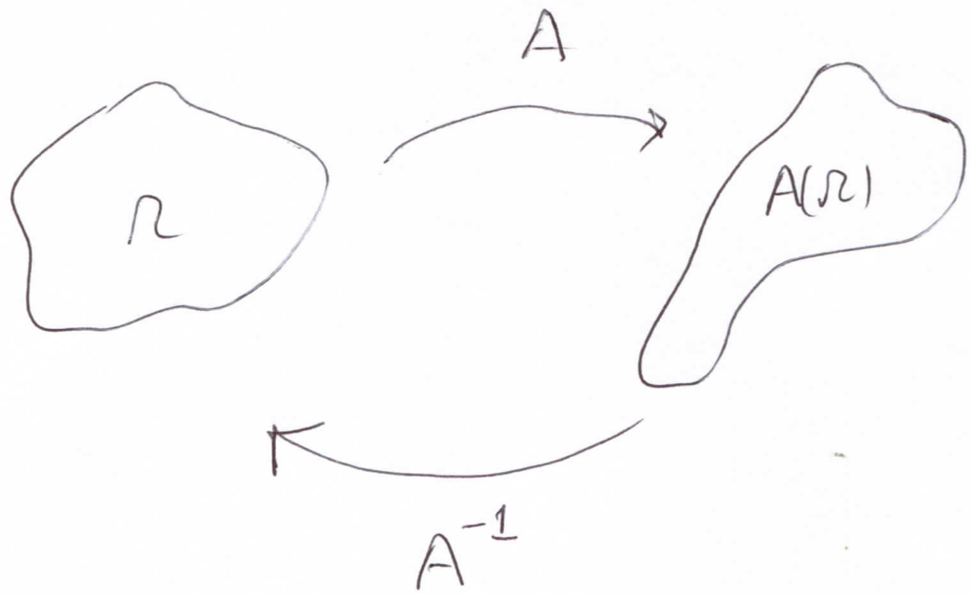
area = |det A|

5.10 (7) implies that the area of $A(C_2)$ is $|\det A|$. Thus C_2, a set of area 1, is transformed by A into $A(C_2)$, a set of area $|\det A|$. More generally, as remarked in 5.1,

$$(\text{area of } A(\Omega)) = (\text{area of } \Omega)\,|\det A|$$

for almost any Ω contained in $\mathbb{R}^2$; thus $|\det A|$ is a **magnification factor** for A.

If A is invertible, its inverse A^{-1} undoes A; in particular, it undoes whatever magnification A performed



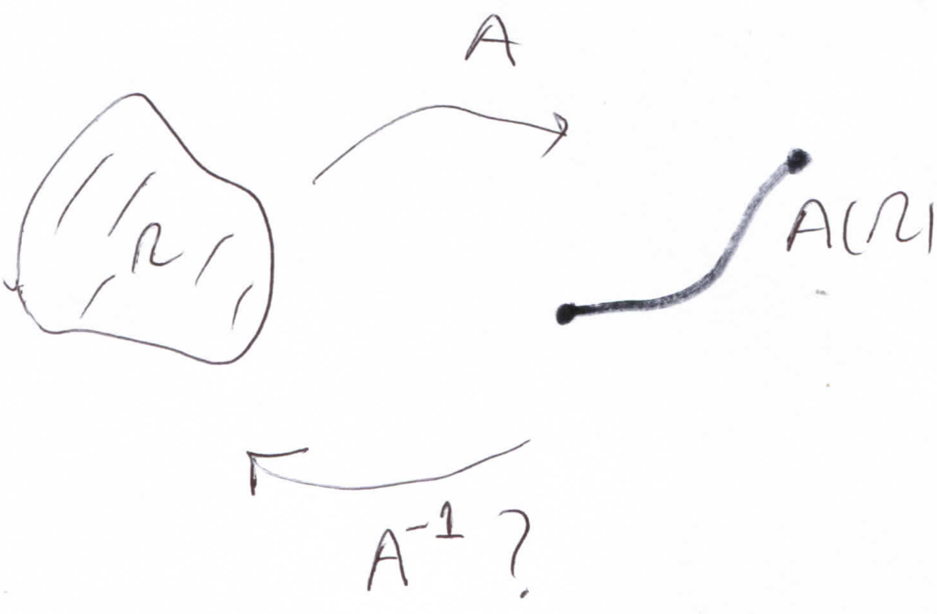
If A deflated Ω, A^{-1} is like a bicycle pump inserted into $A(\Omega)$, inflating it back to the area of Ω.

A^{-1}, if it exists, has its own magnification factor

$$|\det(A^{-1})| = \frac{1}{|\det A|}.$$

We learn to shudder at the thought of dividing by zero, thus $\det(A^{-1})$ is problematic when $\det A = 0$. Applying an A^{-1} appears to require an infinite magnification.

Getting back to our bicycle pump, if $\det A = 0$, then A deflates two-dimensional Ω into nothing; at least nothing with any area.



It doesn't seem plausible that we can inflate something completely crushed; it's like trying to bring a squashed bug back to life.