SECTION VIC:

ORTHOGONAL SETS and BASES

We have defined a pair of vectors being orthogonal. More generally, we would like to define a <u>set</u> of arbitrary size being orthogonal.

DEFINITION 6.29

The set of n-vectors S is **orthogonal** if each pair of vectors in S is orthogonal; that is,

$$\vec{x} \cdot \vec{y} = 0$$

whenever $\vec{x}$ and $\vec{y}$ are two different vectors in S.

Examples 6.30

(a) $\{\vec{i} \equiv (1,0,0),\ \vec{j} \equiv (0,1,0),\ \vec{k} \equiv (0,0,1)\}$

is orthogonal, since

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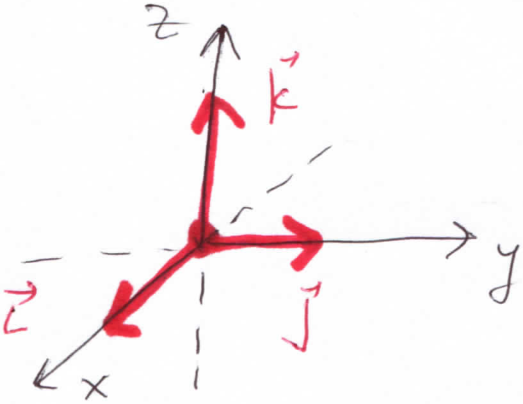
$$\vec{i} \cdot \vec{j} = 0 = \vec{i} \cdot \vec{k} = \vec{j} \cdot \vec{k}$$



$\vec{i}, \vec{j}, \vec{k}$ are popular in physics; note that, for any real a, b, c,

$$(a, b, c) = a\vec{i} + b\vec{j} + c\vec{k}$$

(b) $\left\{ \vec{a} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{d} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7 \\ 1 \end{bmatrix} \right.$

is orthogonal, since $0 = \vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$

$= \ldots$ (6 pairs of dot products to check)

So long as we avoid the trivial vector $\vec{0} \equiv (0, 0, 0, \ldots)$, orthogonality is stronger than linear independence.

THEOREM 6.31

If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} \equiv S$ is an orthogonal set of nontrivial vectors, then S is linearly independent.

Proof: Suppose a linear combination from S equals $\vec{0}$; that is,

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$(*)$ $(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m) = \vec{0}$,

for some real numbers

$c_1, c_2, \ldots, c_m$.

Recall that linear independence is equivalent to $(*)$ occurring only when $c_1 = 0 = c_2 = c_3 = \cdots = c_m$; that is, only the trivial linear combination equals $\vec{0}$.

$(*)$ implies that

$$0 = \vec{v}_1 \cdot \vec{0} = \vec{v}_1 \cdot (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots)$$

$$= c_1(\vec{v}_1 \cdot \vec{v}_1) + c_2(\vec{v}_1 \cdot \vec{v}_2) + \cdots$$

$$= c_1\|\vec{v}_1\|^2,$$ by orthogonality;

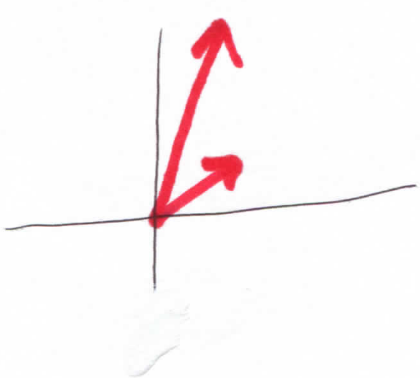
since $\vec{v}_1$ is nontrivial,
$\|\vec{v}_1\|^2 \neq 0$, thus $c_1 = 0$.

The same argument, with $\vec{v}_j$, $j = 2, 3, \ldots, m$ replacing $\vec{v}_1$, shows that

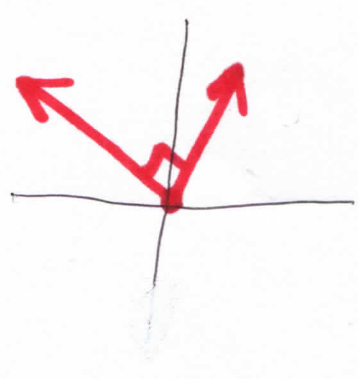
$$c_j = 0 \text{ for } j = 1, 2, 3, \ldots, m,$$

as desired.

Here is the picture in $\mathbb{R}^2$:



linear independence



orthogonality

For a pair of vectors in $\mathbb{R}^2$, linear independence means they are not parallel, while orthogonality means they are perpendicular.

DEFINITION 6.32

If W is a subspace with $\dim(W) = \underline{p}$, then any orthogonal set of $\underline{p}$ nontrivial vectors in W is called an

orthogonal basis

for W.

Note that Theorems 4.56 and 6.31 imply that an orthogonal basis is a basis in the sense of Chapter IV.

Theorems 6.31 and 4.56 show that an orthogonal set is better than a linearly independent set and an orthogonal basis is better than a basis.

Is this superiority substantive? Or is it "so what"?

The remainder of this section will describe three substantial advantages to using orthogonal sets of vectors.

6.33. ORTHOGONAL ADVANTAGE (1).

(extended Pythagorean Theorem)

If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ is an orthogonal set of vectors, then

$$\|\vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_m\|^2 = \|\vec{v}_1\|^2 + \|\vec{v}_2\|^2 + \ldots + \|\vec{v}_m\|^2.$$

Examples 6.34

(a) See Examples 6.30(b).

For any real c_1, c_2, c_3, c_4,

by 6.33,

$$\left\| c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7 \\ 1 \end{bmatrix} \right\|^2 =$$

$$c_1^2 \| (1,2,0,0,0) \|^2 + c_2^2 \| (-2,1,0,0,0) \|^2$$

$$+ c_3^2 \| (0,0,1,0,0) \|^2 + c_4^2 \| (0,0,0,7,1) \|^2 =$$

$$5c_1^2 + 5c_2^2 + c_3^2 + 50c_4^2.$$

(b) $\| (1,-1) + (0,1) \|^2 = \| (1,0) \|^2 = 1$, but

$\| (1,-1) \|^2 + \| (0,1) \|^2 = 2+1 = 3$;

6.33 NEEDS orthogonality.

Given a basis $\{\vec{v}_1, \vec{v}_2, \ldots\}$ for a vector space V, we know that any $\vec{x}$ in V is a linear combination of the basis:

$$(*) \left(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots\right) = \vec{x},$$

for <u>some</u> real numbers $c_1, c_2, c_3, \ldots$

Determing $c_1, c_2, \ldots$ is, in general, a very big deal, as we have all learned from bitter experience; $(*)$ is the vector form of a linear system in the variables

$c_1, c_2, \ldots,$ to be solved (normally) by Gauss-Jordan elimination.

Examples 6.35

(a) Write $(1,2,3,4)$ as a linear combination of

$\{(5,6,7,8), (9,0,1,2), (3,4,5,6), (-7,-8,-9,0)\}$??

We would seek real numbers c_1, c_2, c_3, c_4 so that

$$c_1\begin{bmatrix}5\\6\\7\\8\end{bmatrix} + c_2\begin{bmatrix}9\\0\\1\\2\end{bmatrix} + c_3\begin{bmatrix}3\\4\\5\\6\end{bmatrix} + c_4\begin{bmatrix}-7\\-8\\-9\\0\end{bmatrix} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

This would mean Gauss-Jordan elimination, applied to the augmented matrix

$$\left[\begin{array}{cccc|c} 5 & 9 & 3 & -7 & 1 \\ 6 & 0 & 4 & -8 & 2 \\ 7 & 1 & 5 & -9 & 3 \\ 8 & 2 & 6 & 0 & 4 \end{array}\right]$$

(b) Write $(1,2,3,4)$ as a linear combination of

$$\{\underbrace{(1,0,0,0)}_{\equiv \vec{e}_1}, \underbrace{(0,1,0,0)}_{\equiv \vec{e}_2}, \underbrace{(0,0,1,0)}_{\equiv \vec{e}_3}, \underbrace{(0,0,0,1)}_{\equiv \vec{e}_4}\} ??$$

This is such a nice basis (we called it the "standard basis")

that we might guess
the answer:

$$(1,2,3,4) = 1\vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3 + 4\vec{e}_4.$$

Notice that

$$1 = (1,2,3,4) \bullet \vec{e}_1$$

$$2 = (1,2,3,4) \bullet \vec{e}_2$$

$$3 = (1,2,3,4) \bullet \vec{e}_3$$

$$4 = (1,2,3,4) \bullet \vec{e}_4$$

This technique will work for <u>any</u>
orthogonal basis, except that
we need unit vectors. Thus,

if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ is an orthogonal basis, we will have

$$\vec{x} = \left(\vec{x} \cdot \frac{\vec{v}_1}{\|\vec{v}_1\|}\right)\left(\frac{\vec{v}_1}{\|\vec{v}_1\|}\right) + \left(\vec{x} \cdot \frac{\vec{v}_2}{\|\vec{v}_2\|}\right)\left(\frac{\vec{v}_2}{\|\vec{v}_2\|}\right)$$

$$+ \ldots$$

$$= \left(\frac{\vec{x} \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\right)\vec{v}_1 + \left(\frac{\vec{x} \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\right)\vec{v}_2 + \ldots$$

6.36 ORTHOGONAL ADVANTAGE (2)

If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ is an orthogonal basis for W, then, for any $\vec{x}$ in W,

$$\vec{x} = \left(\frac{\vec{x} \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\right)\vec{v}_1 + \left(\frac{\vec{x} \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\right)\vec{v}_2 +$$

$$\cdots + \left(\frac{\vec{x} \cdot \vec{v}_m}{\|\vec{v}_m\|^2}\right)\vec{v}_m$$

Example 6.37 Write $(5, 0, -3)$ as a linear combination of $\{(1,1,-2), (0,2,1), (5,-1,2)\}$.

CHECK first that the set of vectors is orthogonal; since there are three vectors and $\dim(\mathbb{R}^3) = 3$, the set of vectors is an orthogonal basis for $\mathbb{R}^3$.

We need dot products!

$(5,0,-3) \cdot (1,1,-2) = 11$

$\|(1,1,-2)\|^2 = 6$

$(5,0,-3) \cdot (0,2,1) = -3$

$\|(0,2,1)\|^2 = 5$

$(5,0,-3) \cdot (5,-1,2) = 19$

$\|(5,-1,2)\|^2 = 30$

$$\boxed{(5,0,-3) = \frac{11}{6}(1,1,-2) - \frac{3}{5}(0,2,1) + \frac{19}{30}(5,-1,2)}$$

REMARKS 6.38

We are putting off the proof of 6.36, since it will be seen to be a special case of Orthogonal Advantage (3).

Notice that 6.36 represents a vector $\vec{x}$ as a sum of one-dimensional projections of $\vec{x}$ onto each member of the orthogonal basis

$$\vec{x} = \text{proj}_{\vec{v}_1}(\vec{x}) + \text{proj}_{\vec{v}_2}(\vec{x}) + \cdots$$

(see 6.16 - 6.18)

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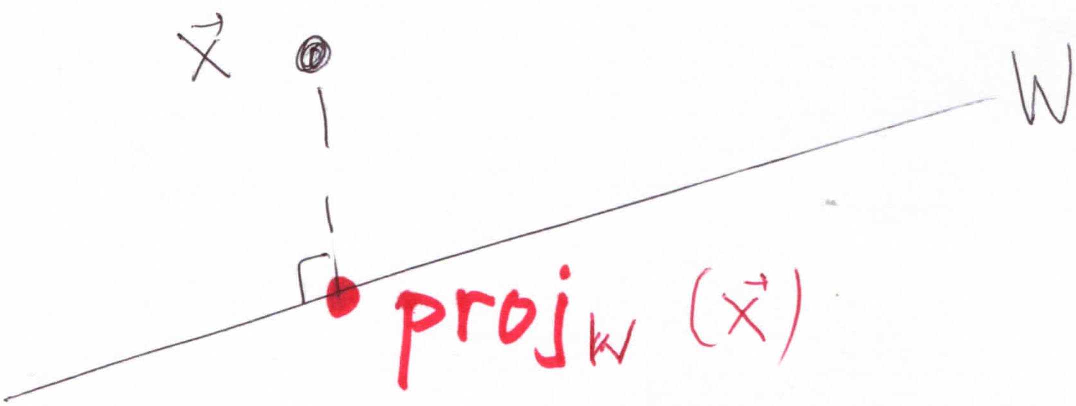
For W a subspace of $\mathbb{R}^n$ and $\vec{x}$ in $\mathbb{R}^n$, we have defined the <u>projection</u> of $\vec{x}$ onto W, denoted

$$\text{proj}_W(\vec{x}), \quad \text{in } 6.13.$$



For $W \equiv \text{span}(\{\vec{b}\})$,

$$\text{proj}_W(\vec{x}) \equiv \text{proj}_{\vec{b}}(\vec{x}) = \left(\frac{\vec{x} \cdot \vec{b}}{\|\vec{b}\|^2}\right)\vec{b}$$

(see Theorem 6.10)

Example 6.39

For a multi-dimensional example, take

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \,\middle|\, x_1, x_2 \text{ real} \right\},$$

the Cartesian plane as a subspace of $\mathbb{R}^3$.

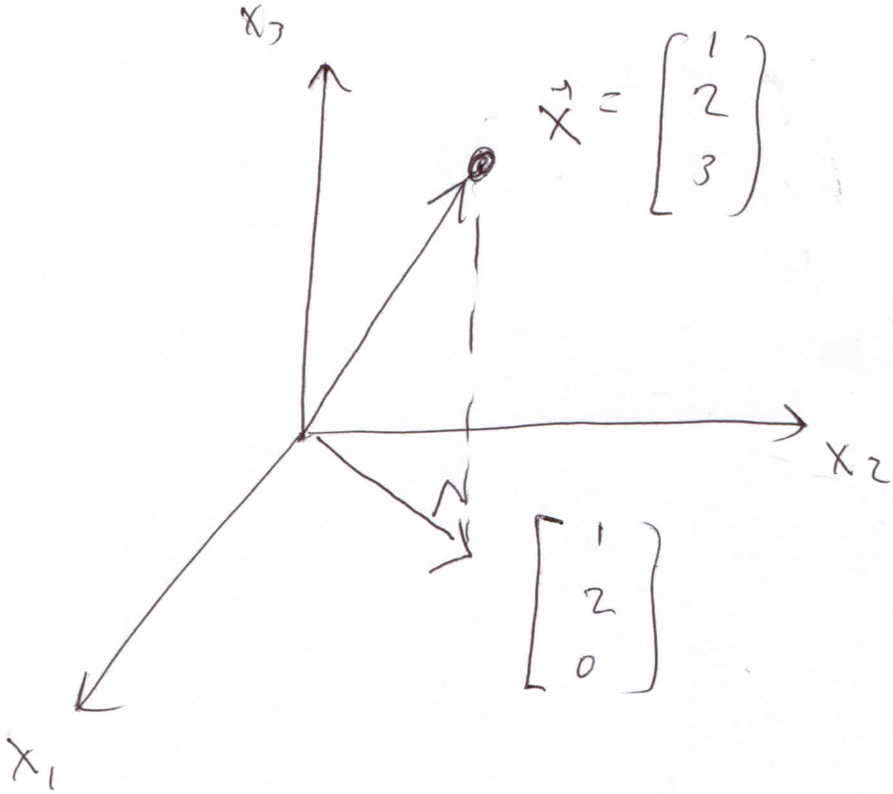
Projection onto W is the oppresive voice of authority, declaring petulantly "No floating!" The x_3 component, representing height above the ground, is deleted, sending you crashing to the ground.

Specifically, let's spoil the fun of

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

by projecting it onto W. The simplest way to get to W is to remove the x_3 coordinate:

$$P_W(\vec{x}) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} ?$$



PROOF that $P_W(\vec{x}) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$:

$$\left(\vec{x} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \perp \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix},$$

for any real x_1, x_2; that is,

$$\left(\vec{x} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right) \perp W.$$

END OF PROOF

NOTE that

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$ is an

orthogonal basis for W, and

$$P_W(\vec{x}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} =$$

$$\left(\frac{\vec{x} \bullet (1,0,0)}{\|(1,0,0)\|^2} \right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \left(\frac{\vec{x} \bullet (0,1,0)}{\|(0,1,0)\|^2} \right) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

This example is meant to motivate the following.

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6.40 ORTHOGONAL ADVANTAGE (3)

If W is a subspace of $\mathbb{R}^n$ and $\{\vec{w}_1, \vec{w}_2, \ldots\}$ is an orthogonal basis for W, then, for any $\vec{x}$ in $\mathbb{R}^n$,

$$P_W(\vec{x}) = \text{proj}_{\vec{w}_1}(\vec{x}) + \text{proj}_{\vec{w}_2}(\vec{x}) + \ldots$$

$$= \left(\frac{\vec{x} \cdot \vec{w}_1}{\|\vec{w}_1\|^2}\right)\vec{w}_1 + \left(\frac{\vec{x} \cdot \vec{w}_2}{\|\vec{w}_2\|^2}\right)\vec{w}_2 + \ldots$$

Proof: $P_W(\vec{x})$ is in W, so

$$P_W(\vec{x}) = c_1\vec{w}_1 + c_2\vec{w}_2 + c_3\vec{w}_3 + \ldots$$

for some real $c_1, c_2, c_3, c_4, \ldots$

By definition of $P_W(\vec{x})$,

$$0 = (\vec{x} - P_W(\vec{x})) \bullet \vec{w},$$

for all $\vec{w}$ in W; in particular,

$$0 = (\vec{x} - P_W(\vec{x})) \bullet \vec{w}_1 =$$

$$(\vec{x} - (c_1\vec{w}_1 + c_2\vec{w}_2 + \dots)) \bullet \vec{w}_1 =$$

$$(\vec{x} \bullet \vec{w}_1) - (c_1(\vec{w}_1 \bullet \vec{w}_1) + c_2(\vec{w}_2 \bullet \vec{w}_1) + \dots)$$

$$= (\vec{x} \bullet \vec{w}_1) - c_1 \|\vec{w}_1\|^2,$$ by

orthogonality, so that

$$(\vec{x} \bullet \vec{w}_1) = c_1 \|\vec{w}_1\|^2 \rightarrow c_1 = \frac{(\vec{x} \bullet \vec{w}_1)}{\|\vec{w}_1\|^2}.$$

For $j = 2, 3, \ldots$, the same argument, with $\vec{w}_j$ replacing $\vec{w}_1$, shows that

$$c_j = \left(\frac{\vec{x} \bullet \vec{w}_j}{\|\vec{w}_j\|^2} \right),$$

so that

$$P_W(\vec{x}) = \left(\frac{\vec{x} \bullet \vec{w}_1}{\|\vec{w}_1\|^2} \right) \vec{w}_1 + \left(\frac{\vec{x} \bullet \vec{w}_2}{\|\vec{w}_2\|^2} \right) \vec{w}_2$$

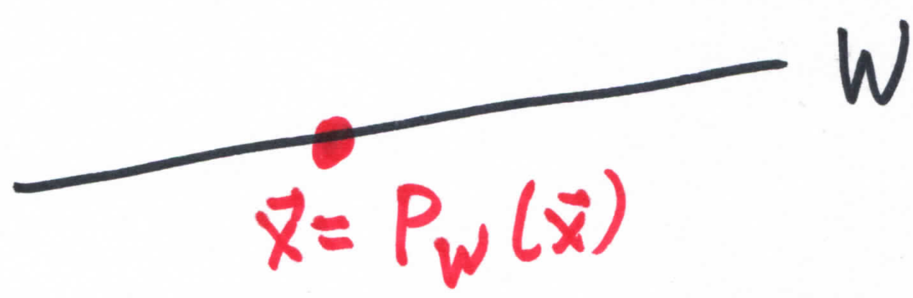
$$+ \cdots,$$

as desired.

REMARKS 6.41

The reader should notice similarity between 6.40 and 6.36. This is because 6.36 is the special case of 6.40 when $\vec{x}$ is itself in W, which is equivalent to

$$\vec{x} = P_W(\vec{x}).$$



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In Example 6.39, $\vec{x}$ being in W means you are already on the ground. The correct answer then to the authority figure of Example 6.39 saying "Drop and give me twenty" is "I can't drop any further, I'm already on the ground."

Example 6.42 Suppose

$W = \text{span}\{(0,1,1,0), (1,0,0,1), (-1,1,-1,1)\}$

Get $P_W((-2,0,5,3))$

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CHECK that

$\{\vec{w}_1 \equiv (0,1,1,0), \vec{w}_2 \equiv (1,0,0,1), \vec{w}_3 \equiv (-1,1,-1,1)\}$

is orthogonal. Then

$\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is an orthogonal

basis for $W \equiv \text{span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$,

thus, with $\vec{x} \equiv (-2,0,5,3)$,

$P_W(\vec{x}) =$

$$\left(\frac{\vec{x} \cdot \vec{w}_1}{\|\vec{w}_1\|^2}\right)\vec{w}_1 + \left(\frac{\vec{x} \cdot \vec{w}_2}{\|\vec{w}_2\|^2}\right)\vec{w}_2 + \left(\frac{\vec{x} \cdot \vec{w}_3}{\|\vec{w}_3\|^2}\right)\vec{w}_3$$

$$= \frac{5}{2}\vec{w}_1 + \frac{1}{2}\vec{w}_2$$

We will conclude this section with a definition that encompasses two desirable properties.

DEFINITION 6.43

An orthogonal set of unit vectors is called an

orthonormal set.

Example and Remark 6.44

(a) $\left\{\frac{1}{\sqrt{2}}(1,1),\ \frac{1}{\sqrt{2}}(1,-1)\right\}$ is an orthonormal set.

(b) Notoce how tidy the formulas in 6.36 and 6.40 are when $\{\vec{w}_1, \vec{w}_2, \ldots\}$ is an orthonormal basis for W:

$$P_W(\vec{x}) = (\vec{x} \cdot \vec{w}_1)\vec{w}_1 + (\vec{x} \cdot \vec{w}_2)\vec{w}_2 + \ldots$$

SECTION VI D: GRAM-SCHMIDT ORTHOGONALIZATION

In the last section we argued the superiority of orthogonal sets over linearly independent sets. This section offers constructive consolation: a particular way to change a linearly independent set into an orthogonal set, without changing span.

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GRAM-SCHMIDT ORTHOGONALIZATION

6.44

Suppose $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$ is a linearly independent set. Construct a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ as follows.

$$\vec{v}_1 \equiv \vec{w}_1$$

$$\vec{v}_2 \equiv \vec{w}_2 - P_{\text{span}(\vec{v}_1)}(\vec{w}_2)$$

$$\vec{v}_3 \equiv \vec{w}_3 - P_{\text{span}(\vec{v}_1, \vec{v}_2)}(\vec{w}_3)$$

$\vdots$

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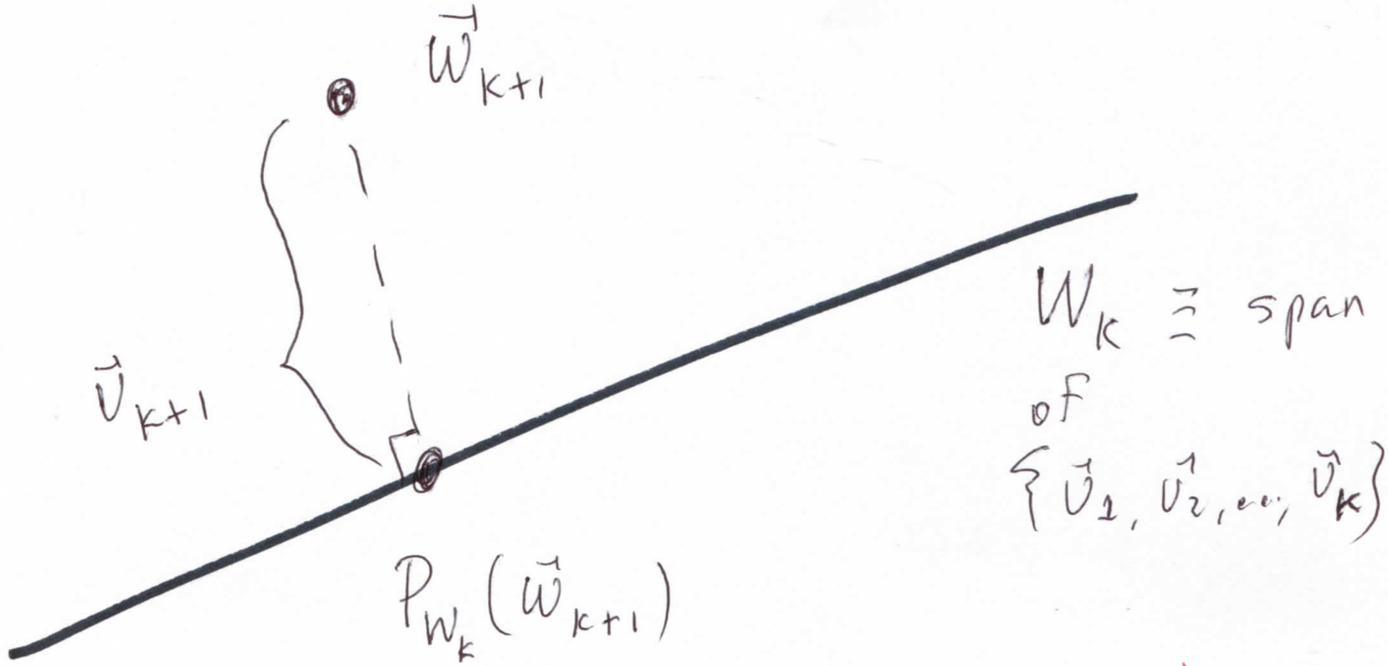
$$\vec{v}_m \equiv \vec{w}_m - P_{\operatorname{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{m-1})}(\vec{w}_m)$$



$(1 \le k \le m-1)$

(see 6.40 for P_{W_k})

Then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ is orthogonal

and

$\operatorname{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j) = \operatorname{span}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_j)$

for $1 \le j \le m$.

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Examples 6.45

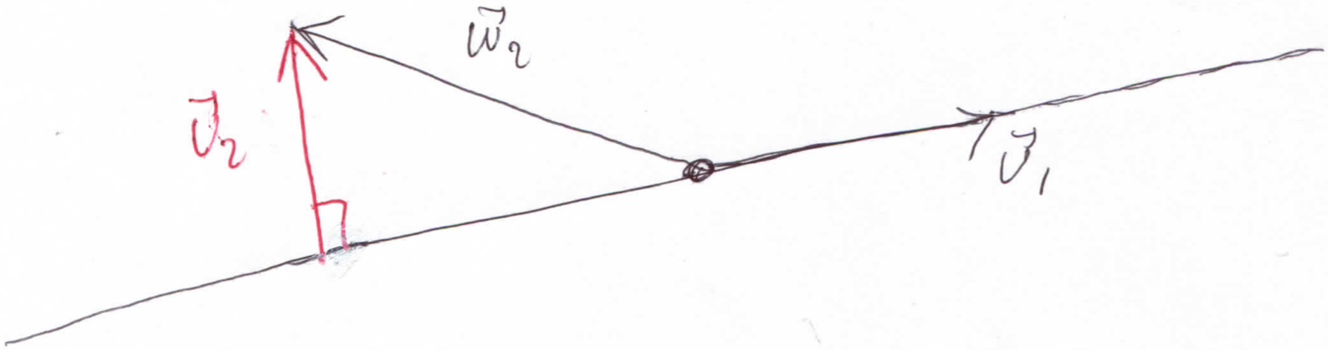
In each of the following, apply Gram-Schmidt.

(a) $\left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$

$\qquad\vec{w}_1 \qquad\qquad \vec{w}_2$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \operatorname{proj}_{(1,-3,0)}\left(\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right)$$



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$$= \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \left(\frac{(-1,1,2) \cdot (1,-3,0)}{\|(1,-3,0)\|^2} \right) \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \left(-\frac{4}{10} \right) \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + \frac{2}{5} \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

$$= \frac{1}{5} \left(\begin{bmatrix} -5 \\ 5 \\ 10 \end{bmatrix} + \begin{bmatrix} 2 \\ -6 \\ 0 \end{bmatrix} \right) = \frac{1}{5} \begin{bmatrix} -3 \\ -1 \\ 10 \end{bmatrix} ;$$

$$\boxed{\{ \vec{v}_1, \vec{v}_2 \} = \left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \frac{1}{5} \begin{bmatrix} -3 \\ -1 \\ 10 \end{bmatrix} \right\}}$$

NOTE that $\vec{v}_1 \cdot \vec{v}_2 = 0$.

(b) $\{(1,0,1,1), (0,-1,2,3), (2,1,0,0)\}$

$$\boxed{\vec{v}_1 \equiv (1,0,1,1)} \leftarrow \text{(SAVE)}$$

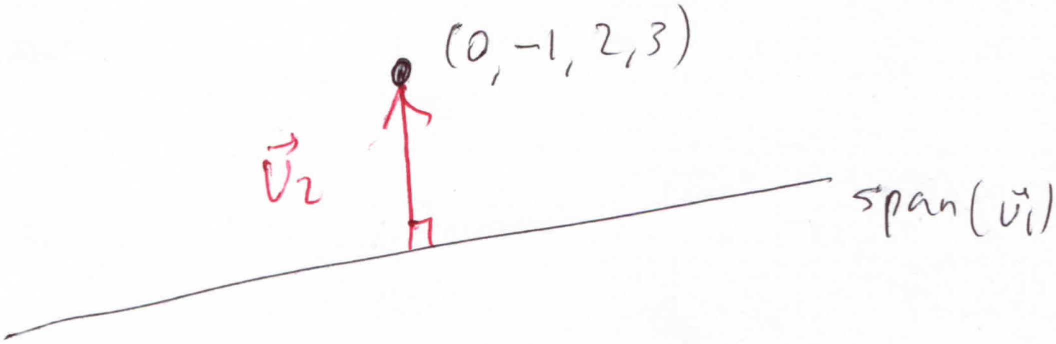
$$\vec{v}_2 \equiv (0,-1,2,3) - \text{proj}_{(1,0,1,1)}(0,-1,2,3)$$

$$= (0,-1,2,3) - \left(\frac{(0,-1,2,3)\cdot(1,0,1,1)}{\|(1,0,1,1)\|^2}\right)(1,0,1,1)$$

$$= (0,-1,2,3) - \left(\frac{5}{3}\right)(1,0,1,1) =$$

$$\frac{1}{3}\left[(0,-3,6,9) - (5,0,5,5)\right] =$$

$$\boxed{\frac{1}{3}(-5,-3,1,4) = \vec{v}_2} \leftarrow \text{(SAVE)}$$



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$$\vec{v}_3 = (2,1,0,0) - P_{span(\vec{v}_1, \vec{v}_2)}(2,1,0,0) =$$

$$(2,1,0,0) - \left[proj_{v_1}(2,1,0,0) + proj_{v_2}(2,1,0,0) \right]$$

$$= (2,1,0,0) - \left[\left(\frac{(2,1,0,0) \cdot (1,0,1,1)}{\|(1,0,1,1)\|^2} \right)(1,0,1,1) \right.$$

$$\left. + \left(\frac{(2,1,0,0) \cdot \cancel{\tfrac{1}{3}}(-5,-3,1,4)}{\| \cancel{\tfrac{1}{3}}(-5,-3,1,4) \|^2} \right) \cancel{\tfrac{1}{3}}(-5,-3,1,4) \right]$$

$$= (2,1,0,0) - \left[\frac{2}{3}(1,0,1,1) + \left(\frac{-13}{51} \right)(-5,-3,1,4) \right]$$

$$= \frac{1}{51}\left[(102,51,0,0) - \left((34,0,34,34) \right.\right.$$

$$\left.\left. + (65,39,-13,-52) \right) \right] =$$

$$\frac{1}{51}(3,12,-21,18) = \underbrace{\boxed{\frac{1}{17}(1,4,-7,6)}}_{\vec{v}_3}$$

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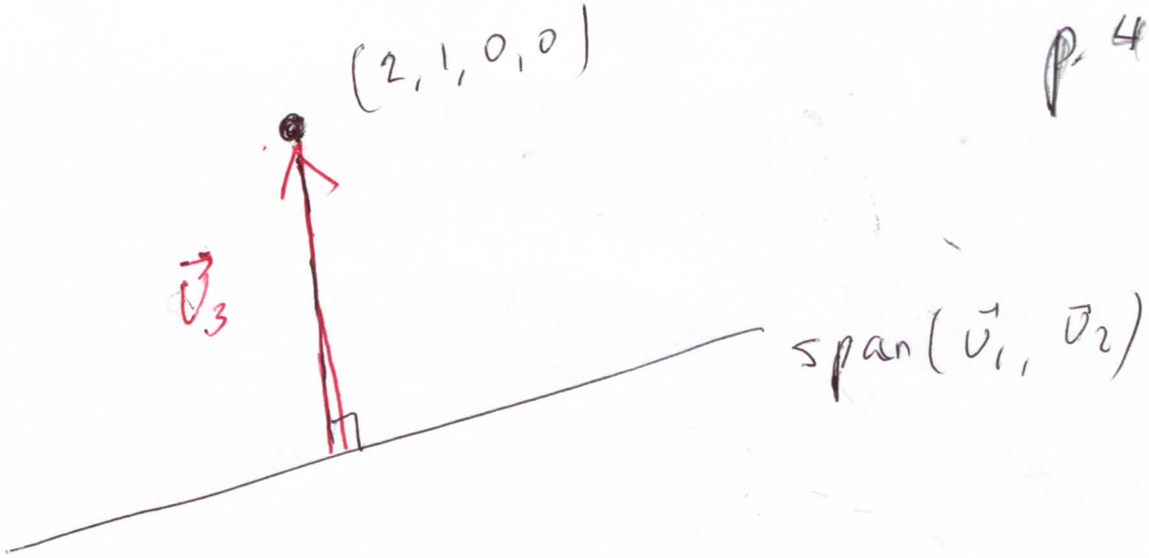

Collecting the red boxes:

$$\boxed{\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{(1,0,1,1),\ \tfrac{1}{3}(-5,-3,1,4),\ \tfrac{1}{17}(1,4,-7,6)\right\}}$$

NOTE that $0 = (1,0,1,1) \cdot (-5,-3,1,4)$
$= (1,0,1,1) \cdot (1,4,-7,6) =$
$(-5,-3,1,4) \cdot (1,4,-7,6)$