SECTION VI F: VECTOR CROSS PRODUCTS

In previous sections we have seen a great deal of dot product. This section will discuss another way of multiplying two vectors together, but it is defined only in $\mathbb{R}^3$.

DEFINITION 6.69

The **cross product**

of $\vec{a} \equiv (a_1, a_2, a_3)$ and

$\vec{b} \equiv (b_1, b_2, b_3)$ is

$$(\vec{a} \times \vec{b}) \equiv \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix},$$

where $\vec{i} \equiv (1, 0, 0)$, $\vec{j} \equiv (0, 1, 0)$,

and $\vec{k} \equiv (0, 0, 1)$; see

Definition 5.5 for

det $\equiv$ <u>determinant</u>

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Examples 6.70

1. $(1, 0, -1) \times (1, 2, 3) =$

$$\det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} =$$

$$\vec{i} \det \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} - \vec{j} \det \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} +$$

$$\vec{k} \det \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \vec{i}\,(0-(-2))$$

$$-\vec{j}\,(3-(-1)) + \vec{k}\,(2-0) =$$

$$2\vec{i} - 4\vec{j} + 2\vec{k} = \boxed{(2, -4, 2)}$$

It is good news that the cross product of two vectors in $\mathbb{R}^3$ is itself in $\mathbb{R}^3$; the dot product of two vectors is only a number, somewhat of a disappointment.

2. $\vec{i} \times \vec{j} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$= \vec{i} \det \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \vec{j} \det \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \vec{k} \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \vec{k}.$

3. $\vec{j} \times \vec{i} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \ldots \; -\vec{k}$

Notice that this multiplication does not commute:

$$\vec{i} \times \vec{j} \neq \vec{j} \times \vec{i}.$$

However, $\vec{i} \times j$ does equal $-(\vec{j} \times \vec{i})$; it's always the case that permuting the vectors multiplies the product by (-1).

PROPOSITION 6.71

For any $\vec{a}, \vec{b}$ in $\mathbb{R}^3$,

$$(\vec{b} \times \vec{a}) = -(\vec{a} \times \vec{b}).$$

Proof: 5.8 (1).

Now that we have two ways to multiply vectors, it seems impossible to resist combining the two.

DEFINITION 6.72

If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are in $\mathbb{R}^3$, the **scalar triple product** of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is

$$\vec{a} \bullet (\vec{b} \times \vec{c})$$

Example 6.73

The scalar triple product of $(1,2,3)$, $(1,0,-1)$, and $(0,1,1)$ is

$$(1,2,3) \cdot \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} =$$

$$(1,2,3) \cdot \left[\vec{i} \det\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} - \vec{j} \det\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + \vec{k} \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

$$= (1,2,3) \cdot (1,-1,1) = 1-2+3$$

$$= 2$$

We will leave it to the reader to calculate that the scalar triple product of $(1,2,3)$, $(1,0,-1)$, and $(0,1,1)$ is

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

In general, here is the easiest way to calculate scalar triple product.

PROPOSITION 6.74

If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are in $\mathbb{R}^3$, then their scalar triple product equals

$$\det \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{bmatrix} .$$

Proof: Denoting $\vec{a} \equiv (a_1, a_2, a_3)$, $\vec{b} \equiv (b_1, b_2, b_3)$, $\vec{c} \equiv (c_1, c_2, c_3)$,

$$\vec{a} \bullet (\vec{b} \times \vec{c}) =$$

$$(a_1, a_2, a_3) \bullet \left(\vec{i} \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - \vec{j} \det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + \vec{k} \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \right)$$

$$= a_1 \det\begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} -$$

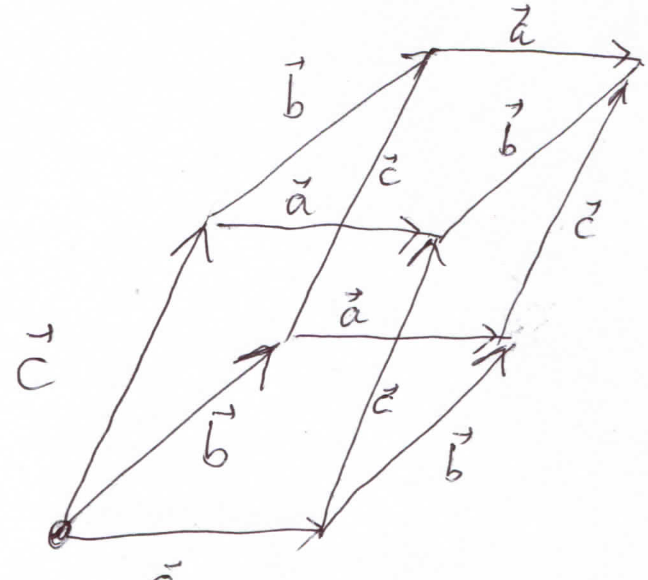
$$a_2 \det\begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \det\begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$$

$$= \det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

To acquire properties of the cross product, we need the following analogue of a parallelogram formed by two vectors in $\mathbb{R}^2$ (see 5.1).

DEFINITION 6.75

If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are in $\mathbb{R}^3$, then the **parallelepiped formed by $\vec{a}, \vec{b}, \vec{c}$** is the set of vectors enclosed by the six parallelograms described below the following sketch.



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Parallelograms enclosing parallelepiped:

1. (~ bottom) vertices
$\vec{0}, \vec{a}, \vec{b}, (\vec{a}+\vec{b})$

2. (~ top) vertices
$\vec{c}, (\vec{a}+\vec{c}), (\vec{b}+\vec{c}), (\vec{a}+\vec{b}+\vec{c})$

3. (~ front) vertices
$\vec{0}, \vec{a}, \vec{c}, (\vec{a}+\vec{c})$

4. (~ back) vertices
$\vec{b}, (\vec{a}+\vec{b}), (\vec{c}+\vec{b}), (\vec{a}+\vec{c}+\vec{b})$

5. (~ left side) vertices
$\vec{0}$, $\vec{b}$, $\vec{c}$, $(\vec{b}+\vec{c})$

6. (~ right side) vertices
$\vec{a}$, $(\vec{b}+\vec{a})$, $(\vec{c}+\vec{a})$, $(\vec{b}+\vec{c}+\vec{a})$.

PROPOSITION 6.76

(Compare to 5.10(7))

The volume of the parallelepiped in Definition 6.75 equals

$$\left| \det \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{bmatrix} \right| .$$

Example 6.77

Find the volume of the parallelepiped formed by the vectors

$(1,2,3)$, $(0,1,0)$, $(1,0,0)$.

SOLUTION: $\left| \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right|$

$= \ldots |(-3)|$

$= 3$.

DEFINITION 6.78

A set of points is **coplanar** if they all lie in the same plane.

Propositions 6.76 and 6.74 imply the following.

PROPOSITION 6.79

If P, Q, R & S are points in $\mathbb{R}^3$, they are coplanar if and only if

$$\vec{PQ} \bullet (\vec{PR} \times \vec{PS}) = 0.$$

Example 6.80 Let

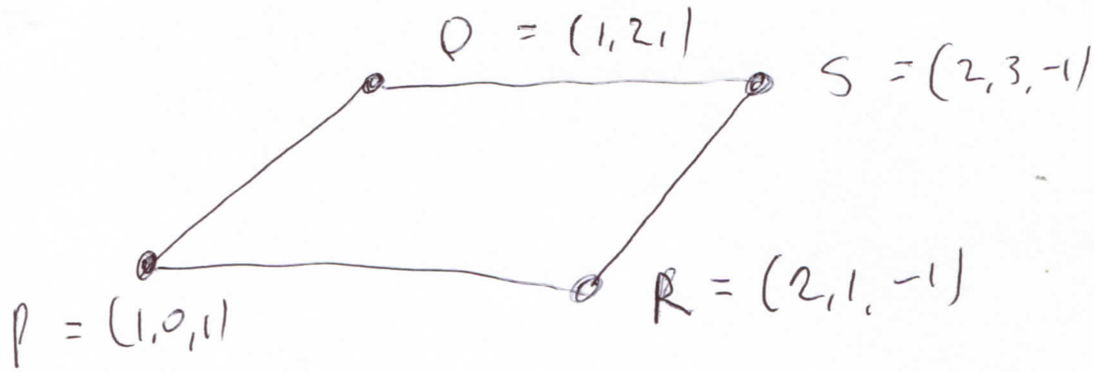
$P \equiv (1,0,1)$, $Q \equiv (1,2,1)$, $R \equiv (2,1,-1)$, $S \equiv (2,3,-1)$.

Then $\vec{PQ} \bullet (\vec{PR} \times \vec{PS}) =$

$$(0,2,0) \bullet \left[(1,1,-2) \times (1,3,-2) \right]$$

$$= \det \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & -2 \\ 1 & 3 & -2 \end{bmatrix} = \ldots 0,$$

so P, Q, R, S are coplanar.



NOTE that $\vec{PQ} = \vec{RS}$ and

$\vec{PR} = \vec{QS}$.

Here are the properties that make the cross product be of interest

THEOREM 6.81

Suppose $\vec{a}$ and $\vec{b}$ are nontrivial vectors in $\mathbb{R}^3$. Then

(1) $(\vec{a} \times \vec{b})$ is $\perp$ to both $\vec{a}$ and $\vec{b}$; and

(2) $\|\vec{a} \times \vec{b}\|$ equals the area of the parallelogram formed by $\vec{a}$ and $\vec{b}$.

Proof: (1) By Proposition 6.74,

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = \det \begin{bmatrix} \vec{a} \\ \vec{a} \\ \vec{b} \end{bmatrix} = 0,$$

by either Proposition 6.76 or 5.10(5). Thus $\vec{a} \perp (\vec{a} \times \vec{b})$, the same argument shows that $\vec{b} \perp (\vec{a} \times \vec{b})$.

(2) Let $\vec{c} \equiv \dfrac{(\vec{a} \times \vec{b})}{\| \vec{a} \times \vec{b} \|}$.

By (1) of this theorem and the fact that $\| \vec{c} \| = 1$, the desired area equals the volume of the parallelepiped formed by $\vec{c}, \vec{a}$, and $\vec{b}$;

by Propositions 6.74 and 6.76, that volume equals

$$\left|\det\begin{bmatrix}\vec{c}\\ \vec{a}\\ \vec{b}\end{bmatrix}\right| = \vec{c}\cdot(\vec{a}\times\vec{b})$$

$$=\left(\frac{1}{\|\vec{a}\times\vec{b}\|}\right)\left((\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})\right)$$

$$=\left(\frac{1}{\|\vec{a}\times\vec{b}\|}\right)\|\vec{a}\times\vec{b}\|^2 = \|\vec{a}\times\vec{b}\|$$

Examples 6.82

Let $P \equiv (0,1,2)$, $Q \equiv (1,1,3)$, $R \equiv (1,0,2)$

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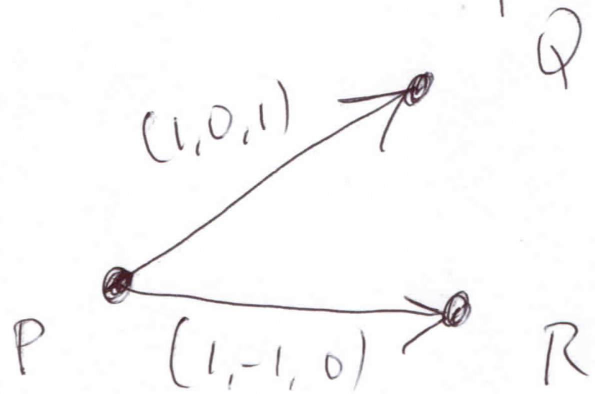
(a) Get the area of the triangle with vertices P, Q, R.

(b) Find a vector $\perp$ to the triangle in (a).

(c) Find the equation of a plane through P, Q, and R.

SOLUTIONS: We need vectors with initial point P



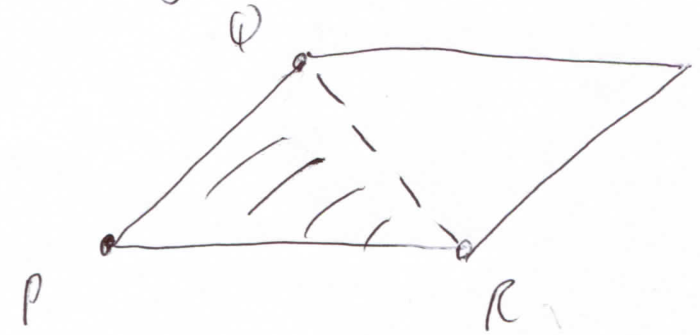
Cross products are crucial to all three question.

$$\vec{PQ} \times \vec{PR} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \vec{i} \det \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \vec{j} \det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$+ \vec{k} \det \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = (1, 1, -1)$$

(a) A triangle is half a parallelogram



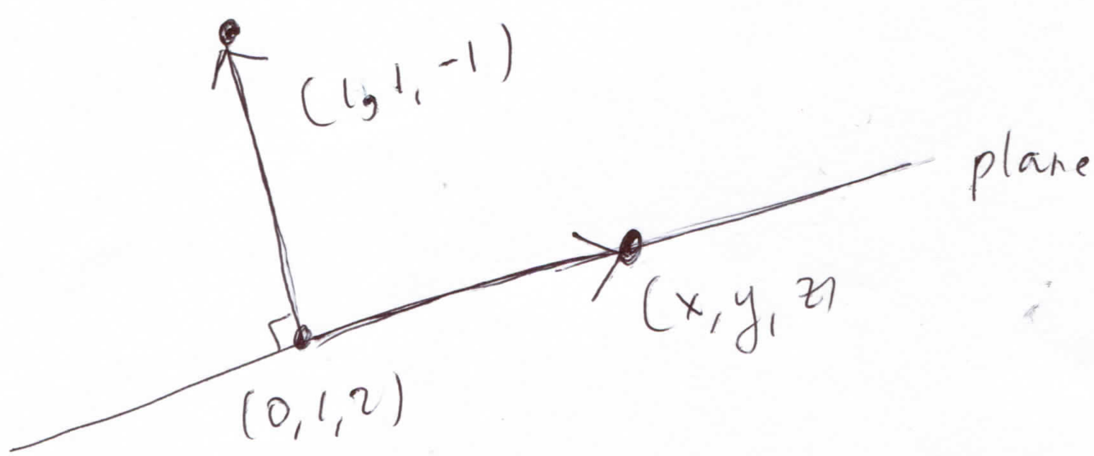
(area of triangle) =

$\frac{1}{2}$(area of parallelogram) =

$\frac{1}{2}\|(1,1,-1)\| = \sqrt{3}/2$

(b) $\boxed{(1,1,-1)}$

(c) Draw a side view of the plane; that is, imagine the plane coming out of the paper



Orthogonality (dot product zero) implies

$$0 = (1,1,-1) \bullet (x-0,\ y-1,\ z-2)$$
$$= x + (y-1) - (z-2) \rightarrow$$

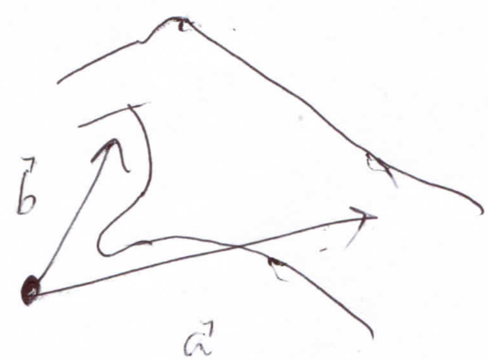
$$\boxed{x + y - z = -1}.$$

REMARKS 6.83

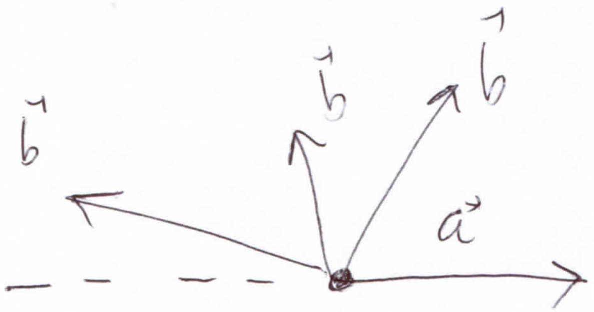
We haven't quite specified the direction of $(\vec{a} \times \vec{b})$ in Theorem 6.81(1). For example, in Examples 6.82(b), $(-1,-1,1) = -(1,1,-1)$ is also

$\perp$ to the triangle
in Examples 6.82(a) and (b)

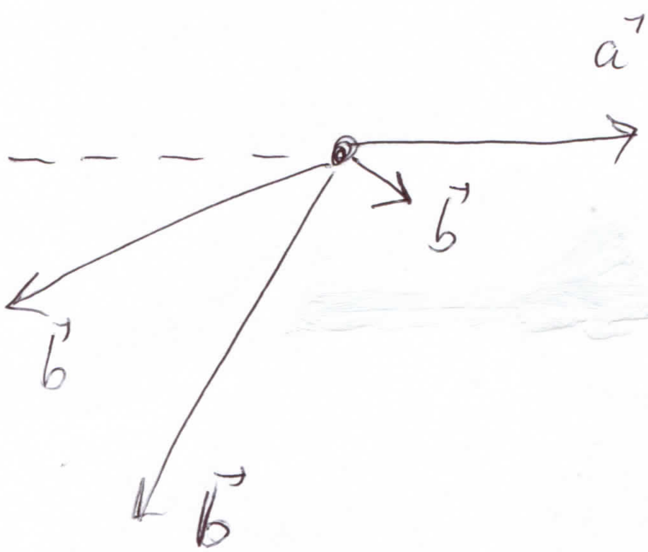
Here is the traditional "right-hand rule" for the direction of $(\vec{a} \times \vec{b})$:
Put the heel of your hand at $\vec{a}$ and your knuckles at $\vec{b}$, then your upright thumb points in the direction of $(\vec{a} \times \vec{b})$.



Ambiguities are possible depending on the size or flexibility of your hand relative to $\vec{a}$ and $\vec{b}$. We prefer the following pictures.



$(\vec{a} \times \vec{b})$ out of paper, towards reader.



$(\vec{a} \times \vec{b})$ into paper, away from reader

Since we've mentioned planes, we should briefly mention lines. All we need is a point and a direction.

DEFINITION 6.84

If $\vec{x}_0$ is a point in $\mathbb{R}^n$ and $\vec{m}$ is a vector in $\mathbb{R}^n$, then the

line thru $\vec{x}_0$ in the direction $\vec{m}$ (see Definitions 6.22)

is the set of all points of the form $\vec{x} = \vec{x}_0 + s\,\vec{m}$ (s real)

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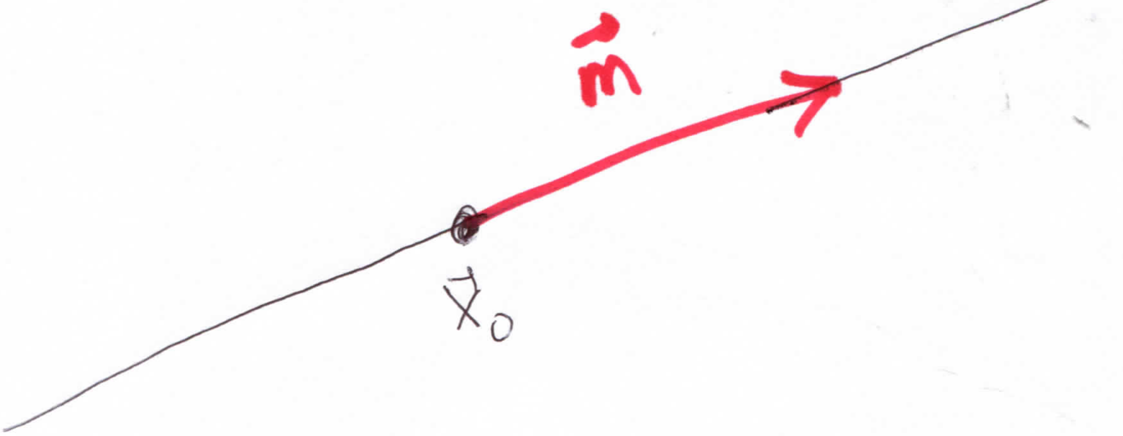
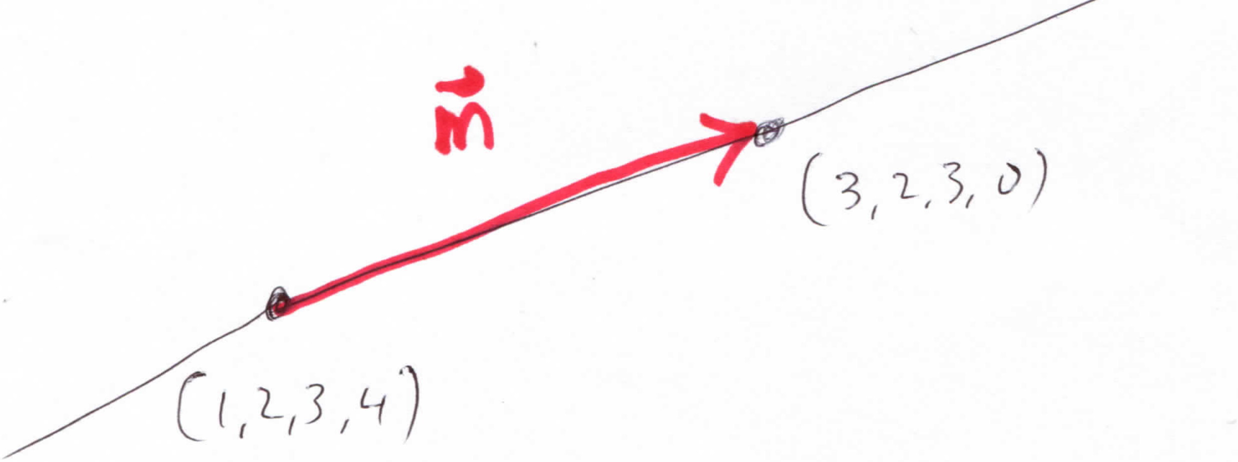


Example 6.85

Find the line through the points $(1,2,3,4)$ and $(3,2,3,0)$.

SOLUTION: We need a direction vector $\vec{m}$



We may choose

$$\vec{m} \equiv ((3-1), (2-2), (3-3), (0-4))$$

$$= (2, 0, 0, -4),$$

then our line is all points of the form

$$\vec{x} = (1, 2, 3, 4) + s(2, 0, 0, -4) \quad (s \text{ real})$$