SECTION VII B: RIGID MOTIONS.

A <u>rigid motion</u> is a function that preserves length, angle, and area. It can be shown that, in $\mathbb{R}^2$, a rigid motion is a combination of translation, rotation and reflection. Translation is not linear, so we will not discuss it here. We would like standard matrices for rotation and reflection. Rotation we will do in this section for only 45 degrees (see Examples 7.11(2), in the last section, for 90 degrees). For rotation by arbitrary angles, we need trigonometry; see Appendix One.

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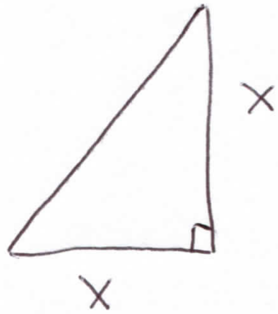
7.12 STANDARD MATRIX for ROTATION BY 45 DEGREES.

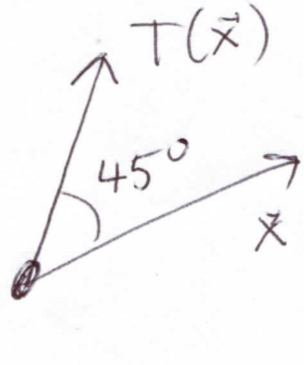
We will derive the matrix in Examples 1.25(3).

We'll need the following right triangle factoid: equal legs is equivalent to a pair of 45 degree angles.

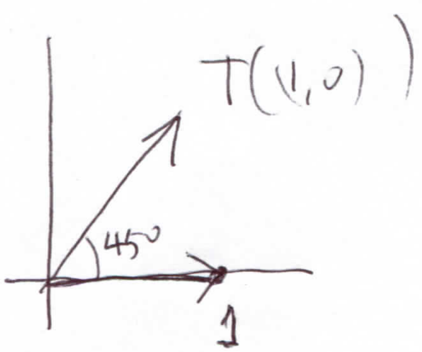


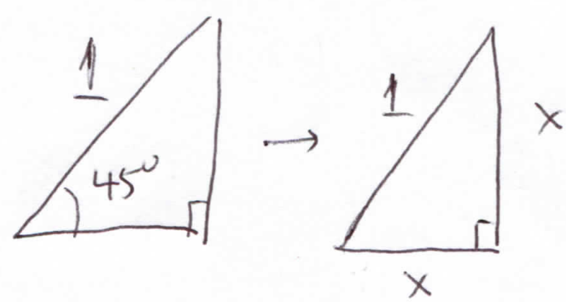
if & only if

Denote by $T: \mathbb{R}^2 \to \mathbb{R}^2$ the function that rotates vectors 45 degrees counter-clockwise: for any $\vec{x}$ in $\mathbb{R}^2$, $T(\vec{x})$ is $\vec{x}$ after being rotated 45 degrees counterclockwise.



As with 90 degree rotation (Examples 7.11 (21)), begin by pretending T is linear and use the proof of Theorem 7.8 to get its standard matrix. We only need $T((1,0))$ & $T((0,1))$:



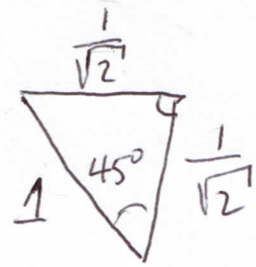


$$\rightarrow \; x^2 + x^2 = 1^2 \;\rightarrow\; x = \frac{1}{\sqrt{2}} \;\rightarrow$$

$$T((1,0)) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Similarly,

$$T((0,1)) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$



Define $A \equiv \left[T((1,0)), T((0,1)) \right]$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

__IF__ T were linear, A would be its standard matrix; that is, we would have

$$T((x_1, x_2)) = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(*)

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix},$$

for any real x_1, x_2.

It remains to prove (*). Fix real x_1, x_2, let $\vec{x} \equiv (x_1, x_2)$, $\vec{y} \equiv (x_1 - x_2, x_1 + x_2)$.

Then

$$\|\vec{y}\|^2 = (x_1 - x_2)^2 + (x_1 + x_2)^2 = 2(x_1^2 + x_2^2)$$

$$= 2\|\vec{x}\|^2, \text{ and}$$

$$\vec{x} \cdot \vec{y} = x_1(x_1 - x_2) + x_2(x_1 + x_2)$$

$$= \|\vec{x}\|^2, \text{ so that}$$

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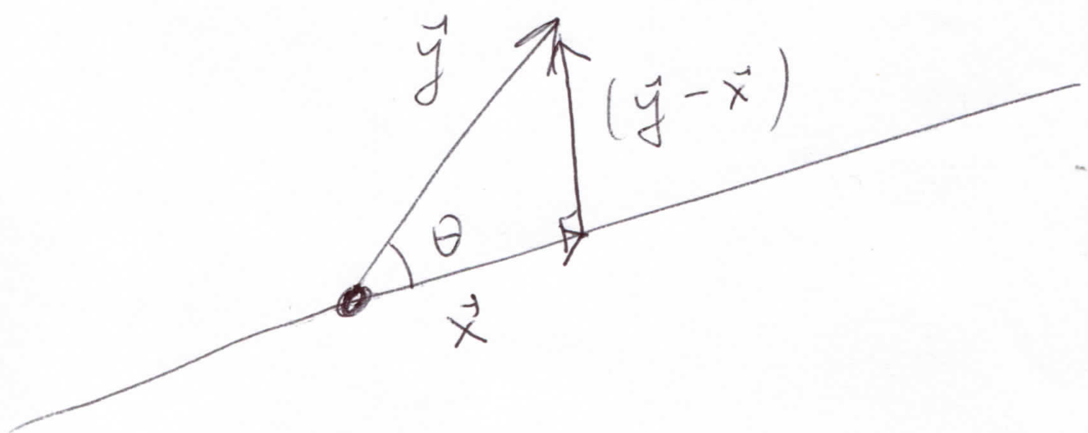
$$\text{proj}_{\vec{x}}(\vec{y}) = \left[\frac{(\vec{x}\cdot\vec{y})}{\|\vec{x}\|^2}\right]\vec{x}$$

$= \vec{x}$ and

$\|\vec{y} - \text{proj}_{\vec{x}}(\vec{y})\|^2 = \|\vec{y} - \vec{x}\|^2 =$

$\|\vec{y}\|^2 + \|\vec{x}\|^2 - 2(\vec{x}\cdot\vec{y}) =$

$2\|\vec{x}\|^2 + \|\vec{x}\|^2 - 2\|\vec{x}\|^2 = \|\vec{x}\|^2;$

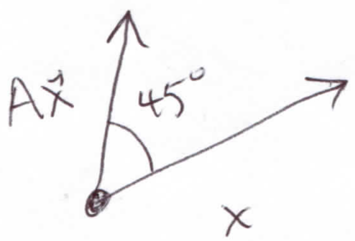


$\vec{x} \perp (\vec{y} - \vec{x})$, $\|\vec{x}\| = \|\vec{y} - \vec{x}\| \rightarrow$

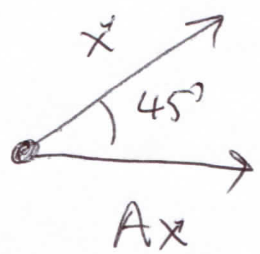
$\theta = 45$ degrees

Thus $\frac{1}{\sqrt{2}}\vec{y} = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(see (*)) is either a 45 degree counterclockwise rotation of $\vec{x}$



or a 45 degree clockwise rotation of $\vec{x}$;



a cross product calculation, as with Examples 7.11(2), shows that $A\vec{x}$ is counterclockwise.

Thus $T((x_1, x_2)) = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

for all real x_1, x_2, as desired.

Note that $A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,

the standard matrix for counterclockwise rotation of 90 degrees.

To get the standard matrix for reflection, we will find it easiest to begin with projection; see Definition 6.16 and Theorem 6.18.

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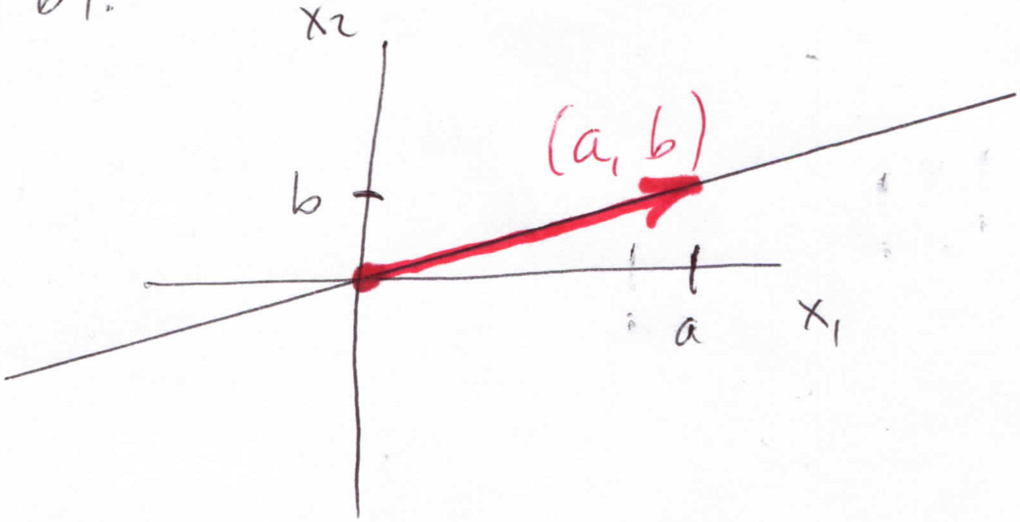
7.13. STANDARD MATRIX for PROJECTION onto a line through the origin.

To get a general formula, we need to specify a direction vector for the line, call it (a, b).



Then our linear transformation is

$$\mathrm{proj}_{(a,b)};$$

for any real $x_1, x_2,$

$$\mathrm{proj}_{(a,b)}(x_1, x_2) = \left[\frac{(x_1, x_2) \bullet (a, b)}{\|(a,b)\|^2}\right](a,b) =$$

$$\left(\frac{x_1 a + x_2 b}{a^2+b^2}\right)(a,b);$$

in particular,

$$\text{proj}_{(a,b)}(1,0) = \left(\frac{a}{a^2+b^2}\right)\begin{bmatrix} a \\ b \end{bmatrix},$$

$$\text{proj}_{(a,b)}(0,1) = \left(\frac{b}{a^2+b^2}\right)\begin{bmatrix} a \\ b \end{bmatrix},$$

so that, by the proof of Theorem 7.8,

$$P \equiv \frac{1}{a^2+b^2}\begin{bmatrix} a^2 & ba \\ ab & b^2 \end{bmatrix} \quad (7.14)$$

is the standard matrix for projection (in $\mathbb{R}^2$) onto the line through $(0,0)$ and (a,b).

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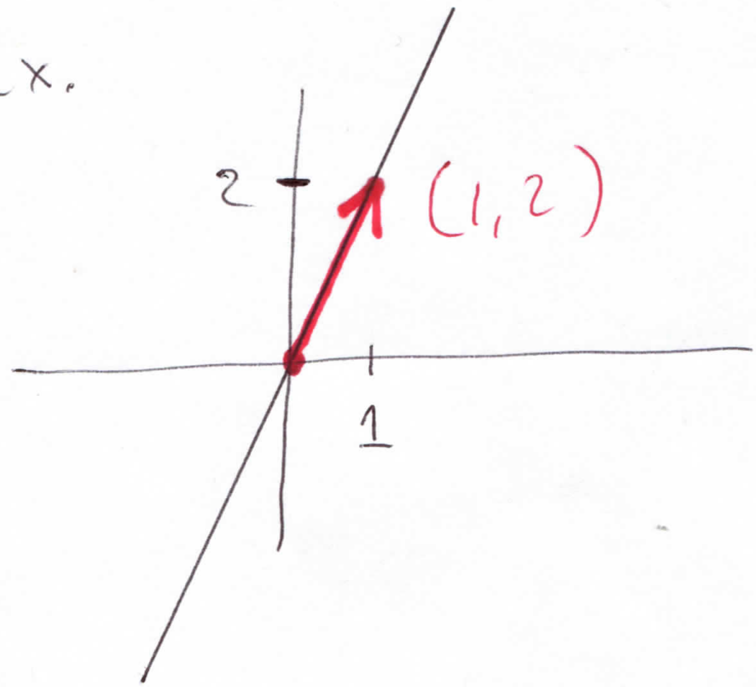
Example 7.15

Get the standard matrix for projection onto the line $y = 2x$.



This is $\text{proj}_{(1,2)}$.

We could ignore (7.14):

$$\text{proj}_{(1,2)}(1,0) = \left[\frac{(1,0)\cdot(1,2)}{\|(1,2)\|^2}\right]\begin{bmatrix}1\\2\end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix}1\\2\end{bmatrix};$$

$$\text{proj}_{(1,2)}(0,1) = \left[\frac{(0,1)\cdot(1,2)}{\|(1,2)\|^2}\right]\begin{bmatrix}1\\2\end{bmatrix}$$

$$= \frac{2}{5}\begin{bmatrix}1\\2\end{bmatrix},$$ thus

$$P = \begin{bmatrix}1/5 & 2/5\\2/5 & 4/5\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$$

is the desired matrix.

OR, we could've used (7.14), with $(a, b) \equiv (1, 2)$:

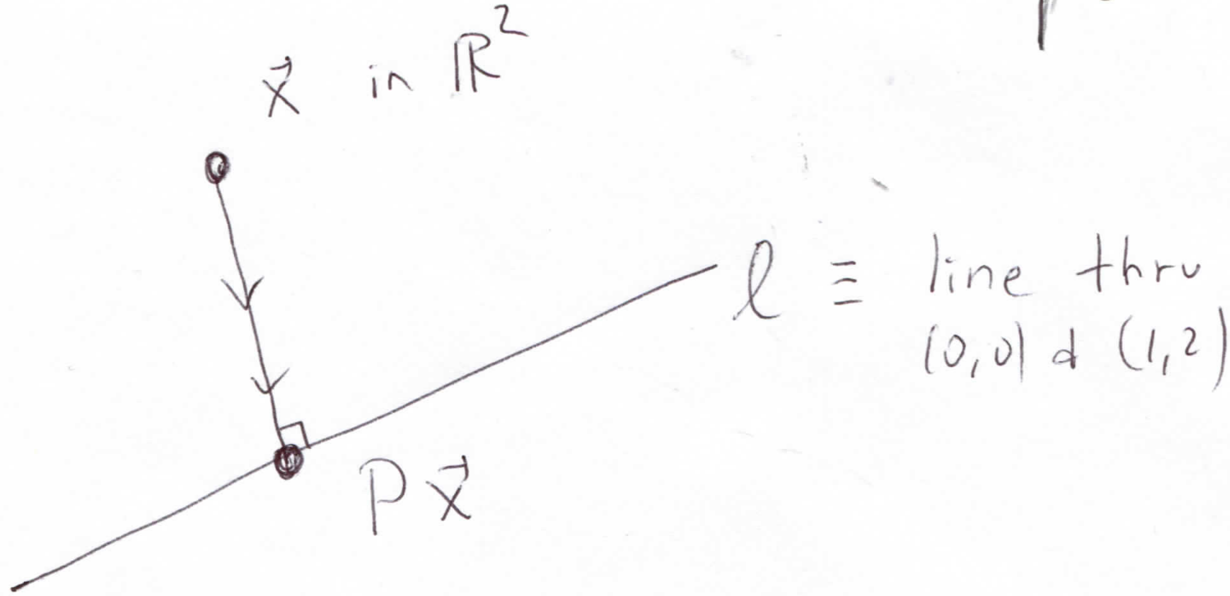
$$P = \frac{1}{1^2 + 2^2} \begin{bmatrix} 1^2 & 2 \cdot 1 \\ 1 \cdot 2 & 2^2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

REMARKS 7.16

We invite the reader to calculate P^2, where P is from the previous example.

If you think of P geometrically, you should know in advance that

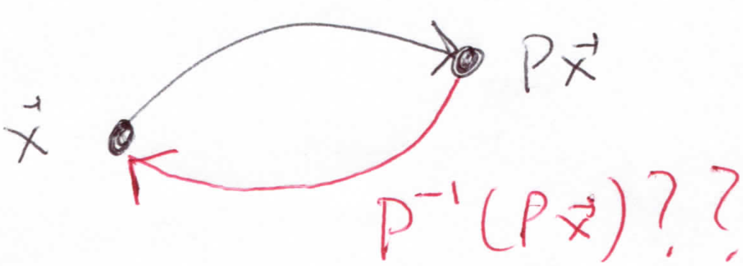
$$P^2 = P.$$



Since $(P\vec{x})$ is on the line ℓ, $P^2\vec{x} \equiv P(P\vec{x})$ must equal $(P\vec{x})$; <u>any</u> point on ℓ equals its projection onto ℓ; dropping to the earth means standing still, if I'm already on the earth.

Another question that could be considered algebraically or geometrically:

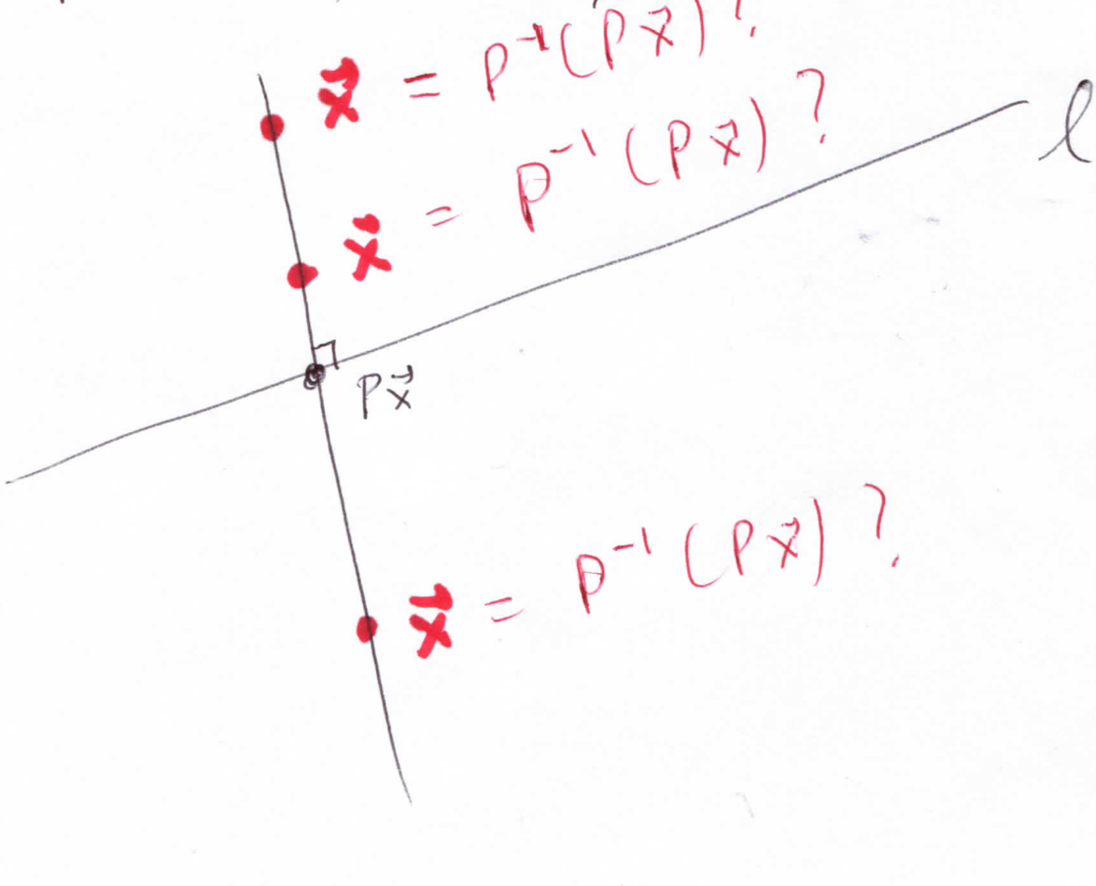
Is P invertible?



P projects all of $\mathbb{R}^2$ onto the line ℓ, denoted

$$P: \mathbb{R}^2 \to \ell.$$

P^{-1}, if it existed, would map ℓ onto all of $\mathbb{R}^2$.
As mentioned in Ch. V (Remarks 5.15) this is like inflating a completely squashed bug; each point $P\vec{x}$ on ℓ has infinitely many choices for $P^{-1}(P\vec{x}) = \vec{x}$:



Algebraically, there are many ways to show that P, from Example 7.15, is not invertible.

We could take its determinant

$$\det P = \frac{1}{25}(1 \cdot 4 - 2 \cdot 2) = 0;$$

see Theorem 5.12.

We could calculate its null space

$$\mathcal{N}(P) = \{(x, -\tfrac{1}{2}x) \mid x \text{ is real}\}$$

Since $\mathcal{N}(P)$ is nontrivial, P is not invertible; see Theorem 5.12.

Or we could argue as follows. If P had an inverse P^{-1}, then $P^2 = P \to$

$$P = P^{-1}(P^2) = P^{-1}P = I.$$

7.16 REFLECTION

through a line.

Here's the picture, for ℓ a line through the origin.

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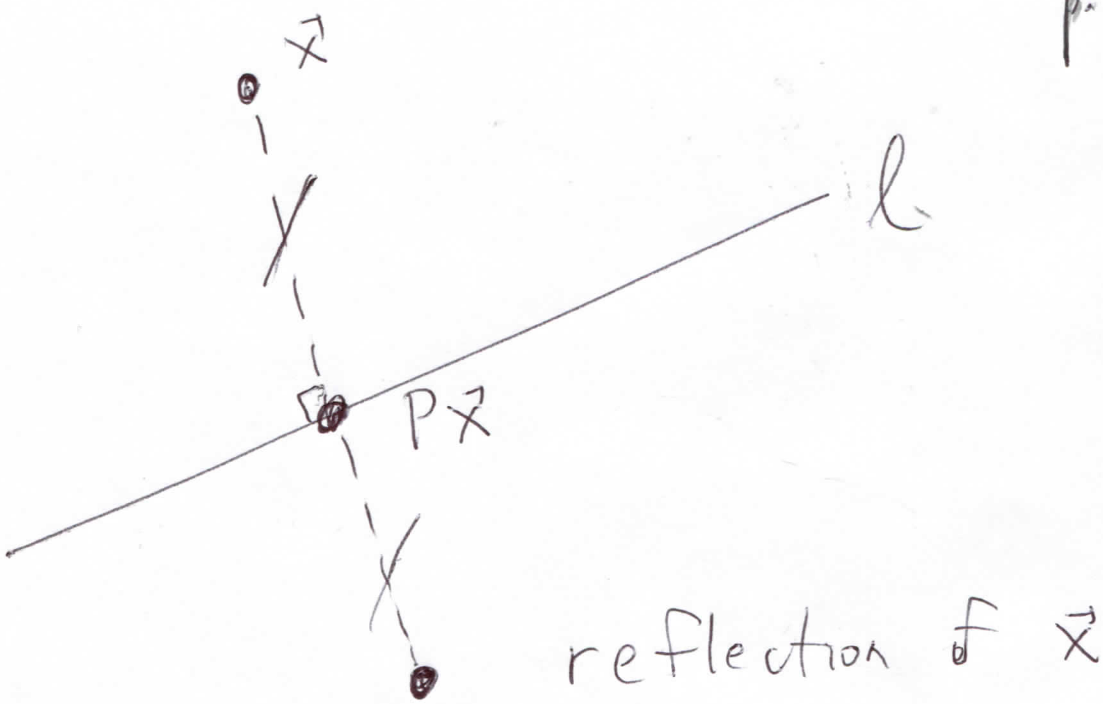


Letting P be as in (7.14), R the standard matrix for reflection, we have

$$(\vec{x} - P\vec{x}) = (P\vec{x} - R\vec{x}),$$

or $$R\vec{x} = 2P\vec{x} - \vec{x}.$$

DEFINITION 7.17

The **reflection** of $\vec{x}$ in $\mathbb{R}^2$ through a line ℓ through the origin $\equiv$

$$\left[2\,\mathrm{proj}_{\ell}(\vec{x}) - \vec{x}\right]$$

PROPOSITION 7.18

If ℓ and P are as in (7.14), then

$$R \equiv \frac{1}{a^2+b^2}\begin{bmatrix} (a^2-b^2) & 2ab \\ 2ab & (b^2-a^2) \end{bmatrix}$$

is the standard matrix for reflection through ℓ.

Proof:

$$(2P - I) =$$

$$\frac{1}{a^2+b^2}\left(\begin{bmatrix} 2a^2 & 2ab \\ 2ab & 2b^2 \end{bmatrix} - \begin{bmatrix} (a^2+b^2) & 0 \\ 0 & (a^2+b^2) \end{bmatrix}\right)$$

$$= \frac{1}{a^2+b^2}\begin{bmatrix} (a^2-b^2) & 2ab \\ 2ab & (b^2-a^2) \end{bmatrix}$$

Example 7.19

Find the standard matrix for reflection through $y = 2x$.

We have already gotten

$$P = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

so by definition of reflection,

$$R = 2\left(\frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\right) - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$$

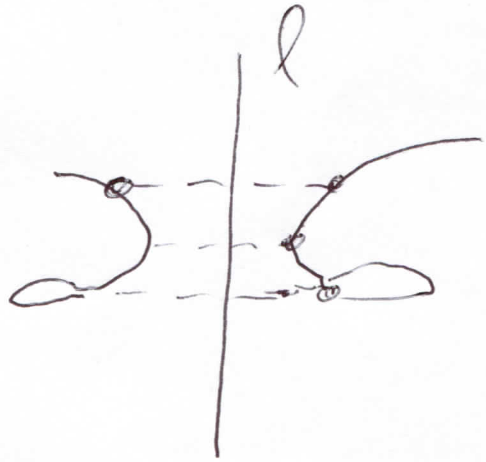
OR, we could have used Proposition 7.18, with $(a,b) \equiv (1,2)$:

$$R = \frac{1}{1^2+2^2}\begin{bmatrix} (1^2-2^2) & 2\cdot 1\cdot 2 \\ 2\cdot 1\cdot 2 & (2^2-1^2) \end{bmatrix}$$

By geometry (check with algebra), $R^2 = I$ & $R^{-1} = R$.

REMARK 7.20

For reflection through ℓ, we think of ℓ as a mirror and $R\vec{x}$ as the reflection of $\vec{x}$ in a mirror.



Here we've reflected each point on a nose.

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Examples 7.21

(a) Find the standard matrix for reflection through $y = 2x$, followed by a 45 degree counterclockwise revolution.

(b) SAME as (a), but in the opposite order.

SOLUTIONS:

Let $A \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ (rotation)

$R \equiv \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$ (reflection)

(a) $\vec{x} \mapsto R\vec{x} \mapsto A(R\vec{x})$

$= (AR)\vec{x}$, so our standard

matrix is

$$AR = \frac{1}{5\sqrt{2}}\begin{bmatrix} -7 & 1 \\ 1 & 7 \end{bmatrix}$$

(b) $\vec{x} \mapsto A\vec{x} \mapsto R(A\vec{x})$

$= (RA)\vec{x}$, so we want

$$RA = \frac{1}{5\sqrt{2}}\begin{bmatrix} 1 & 7 \\ 7 & -1 \end{bmatrix}$$

NOTE that (a) and (b) have different answers; see Examples 1.23.

SECTION VIIC:

MORE EXAMPLES,

including

DIFFERENCE EQUATIONS

A difference equation (see Section IC) is a linear transformation applied repeatedly. We will begin this section with derivations of the matrices in Section IC. Then we will consider (3×3) matrices that do things to three beakers of water. We will conclude with the Fibonacci numbers and their construction with a difference equation.

Examples 7.22

We will contruct the matrices in Examples 1.25. The rotation matrices in Examples 1.25(2) and (3) we have already considered, in 7.11(2) and 7.12.

(a) For T the linear transformation that permutes components of vectors in $\mathbb{R}^2$, as in Examples 1.25(1), we could use the proof of Theorem 7.8 to get the standard matrix:

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\rightarrow$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$$

$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ is the standard matrix

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OR we could write

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 + x_2 \\ x_1 + 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{standard matrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b) In Examples 1.25(4), we want a matrix A so that

$$\vec{x}_{k+1} = A\vec{x}_k \quad , \quad (k = 0, 1, 2, \dots)$$

where $\vec{x}_n \equiv \begin{bmatrix} r_n \\ f_n \end{bmatrix}$, $(n = 0, 1, 2, \dots)$

$$r_{k+1} = 100 r_k - 360 f_k$$

$$f_{k+1} = 4 f_k$$

so

$$\begin{bmatrix} 100 r_k & - 360 f_k \\ 0 & + 4 f_k \end{bmatrix} = \begin{pmatrix} \text{pull out} \\ \text{coefficients} \end{pmatrix}$$

$$\begin{bmatrix} 100 & -360 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} r_k \\ f_k \end{bmatrix}$$

$$\rightarrow \quad A = \begin{bmatrix} 100 & -360 \\ 0 & 4 \end{bmatrix}$$

We could also get A with the standard basis

$$\left\{ \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

as in the proof of

Theorem 7.8.

$\vec{e}_1$ means 1 rabbit, no foxes.

This means no foxes next

year and 100 rabbits next

year; that is,

$$A\vec{e}_1 = \begin{bmatrix} 100 \\ 0 \end{bmatrix}.$$

$\vec{e}_2$ means no rabbits and 1 fox.

This means 4 foxes next

year and (-360) rabbits next

year (the fox ate 360

nonexistent rabbits; we must indulge in science fiction). Thus

$$A\vec{e}_2 = \begin{bmatrix} -360 \\ 4 \end{bmatrix},$$

$$A = \begin{bmatrix} 100 & -360 \\ 0 & 4 \end{bmatrix},$$

putting those columns together.

(c) See Examples 1.25(5). We want a matrix A so that, if

$$\vec{x} \equiv \begin{bmatrix} \begin{pmatrix} \text{number of Moonorgs} \\ \text{not on the moon now} \end{pmatrix} \\ \begin{pmatrix} \text{number of Moonorgs} \\ \text{on the moon now} \end{pmatrix} \end{bmatrix},$$

then $A\vec{x} =$

$$\begin{bmatrix} \begin{pmatrix} \text{number of Moonorgs} \\ \text{not on the moon next year} \end{pmatrix} \\ \begin{pmatrix} \text{number of Moonorgs} \\ \text{on the moon next year} \end{pmatrix} \end{bmatrix}.$$

Again using the proof of Theorem 7.8, we look at

$\vec{e}_1 \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; imagine

1 Moonorg off the moon,

no Moonorgs on the moon.

From the description at the beginning of Examples 1.25 (5) " 20% of Moonorgs on the moon leave the moon, while 10% of Moonorgs not on the moon move to the moon," next year we will have 0.9 Moonorgs off the moon and 0.1 Moonorgs on the moon; that is,

$$A\vec{e}_1 = \begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix}.$$

Argue similarly for $\vec{e}_2 \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; no Moonorgs off the moon and 1 Moonorg on the moon implies that, a year later, there will be 0.2 Moonorgs off the moon, leaving 0.8 Moonorgs on the moon; that is,

$$A\vec{e}_2 = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}.$$

Moosh together those two columns $A\vec{e}_1$ & $A\vec{e}_2$:

$$A = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$$

Examples 7.23

In each of the following, get the standard matrix for the linear transformation T.

(a) $T: \mathbb{R}^3 \to \mathbb{R}^3$

defined to be the orthogonal projection onto the plane $\{(x_1, x_2, 0) \mid x_1, x_2 \text{ real}\}$

SOLUTION: Using the proof of Theorem 7.8 again,

$$T(\vec{e}_1) \equiv T((1,0,0)) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(\vec{e}_2) \equiv T((0,1,0)) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

since $\vec{e}_1$ and $\vec{e}_2$ are in the specified plane.

We claim that

$$T(\vec{e}_3) \equiv T\big((0,0,1)\big) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \equiv \vec{0},$$

since

$$(\vec{e}_3 - \vec{0}) \perp (x_1, x_2, 0)$$

for all real x_1, x_2.

Putting together the three columns $T(\vec{e}_1)$, $T(\vec{e}_2)$, $T(\vec{e}_3)$ gives us

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

as the standard matrix.

(b) Imagine three beakers that want to hold water.

Every hour, the following operations are performed on these beakers, in the order given.

(i) Half the contents of Beaker One are poured into Beaker Three.

(ii) One third of the contents of Beaker Three are poured into Beaker Two.

(iii) All the contents of Beaker One are destroyed.

For $k = 0, 1, 2, \ldots,$ let

$1_k \equiv$ number of grams of water in Beaker One k hours after noon, Jan. 1, 2017;

$2_k \equiv$ same, for Beaker Two

$3_k \equiv$ same, for Beaker Three,

$$\vec{x}_k \equiv \begin{bmatrix} 1_k \\ 2_k \\ 3_k \end{bmatrix}$$

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Find a matrix A

so that

$$\vec{x}_{k+1} = A\vec{x}_k , \quad k = 0, 1, 2, \ldots$$

SOLUTION: Let's track what happens to $\vec{e}_1, \vec{e}_2, \vec{e}_3$ (see proof of Theorem 7.8):

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{(i)} \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} \xrightarrow{(ii)} \begin{bmatrix} 1/2 \\ 1/6 \\ 1/3 \end{bmatrix} \xrightarrow{(iii)} \begin{bmatrix} 0 \\ 1/6 \\ 1/3 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{(i)} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{(ii)} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{(iii)} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{(i)} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{(ii)} \begin{bmatrix} 0 \\ 1/3 \\ 2/3 \end{bmatrix} \xrightarrow{(iii)} \begin{bmatrix} 0 \\ 1/3 \\ 2/3 \end{bmatrix}$$

Paste those columns together:

$$A = [A\vec{e}_1 \ A\vec{e}_2 \ A\vec{e}_3]$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1/6 & 1 & 1/3 \\ 1/3 & 0 & 2/3 \end{bmatrix}$$

Example 7.24

Consider the following population model for a primitive organism. It takes one day to mature, then produces one offspring every day.

Assuming no deaths,
we'd like to know, for
arbitrary $k = 0, 1, 2, \ldots$, the
number of organisms k days
after Jan. 1, 2017, if one
organism is born Jan. 1, 2017

Denote $Y \equiv$ immature, $O \equiv$ mature.

Let's devote the next page
to following the population
growth.

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						Y	
					O	O	
				Y			
			O			O	
					Y		
				O		Y	
		Y			O	O	
						O	
					Y		
Y	O			O		Y	
					O		
			Y			O	
		O				Y	
					O		
				Y		O	
			O			O	
					Y		
				O		Y	
					O		
						O	
							13 + 8 =
1	1	2	3	5	8	13	21

Notice that each population is the sum of the previous two:

$1+1=2$, $1+2=3$, $2+3=5$, $3+5=8$, $5+8=13$, etc.

DEFINITION 7.25

The **Fibonacci numbers** are defined as follows.

$F_1 = 1 = F_2$, and, for $N = 3, 4, 5, \ldots$

$$F_N = F_{(N-1)} + F_{(N-2)}$$

In the population model of Example 7.24, $F_{(N-1)}$ is yesterday's population surviving to today and $F_{(N-2)}$ is yesterday's mature population (it had a day to mature) producing offspring.

In Example 7.24, we calculated

$$F_3 = F_2 + F_1 = 1+1 = 2$$
$$F_4 = F_3 + F_2 = 2+1 = 3$$
$$F_5 = F_4 + F_3 = 3+2 = 5$$

$F_6 = F_5 + F_4 = 5 + 3 = 8$

$F_7 = F_6 + F_5 = 8 + 5 = 13$

$F_8 = F_7 + F_6 = 13 + 8 = 21$

$F_9 = F_8 + F_7 = 21 + 13 = 34$

$\vdots$

The definition is recursive;

F_N is defined by $F_{(N-1)}$ & $F_{(N-2)}$.

To calculate $F_{1,000,000}$, we first need $F_1, F_2, \ldots, F_{999,999}$

As a first step to getting an explicit formula for F_N, we will now show how $\{F_N\}_{N=1}^{\infty}$ arises as a solution of a difference equation.

For $k = 0, 1, 2, 3, \ldots$ define

$$\vec{X}_k \equiv \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix} \qquad (F_0 \equiv 0)$$

Then

$$\vec{X}_{k+1} = \begin{bmatrix} F_{k+1} \\ F_{k+2} \end{bmatrix} = \begin{bmatrix} F_{k+1} \\ F_k + F_{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vec{X}_k \, ;$$ thus, for $N = 1, 2, 3, \ldots,$

$$\begin{bmatrix} F_N \\ F_{N+1} \end{bmatrix} = \vec{X}_N = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^N \vec{X}_0$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^N \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

What remains is explicit formulas for powers of matrices (more generally, see Theorem 1.29 and Remarks 1.30; Examples 1.25(4) will also be of interest). This will be the subject of the next chapter.