CHAPTER VIII: EIGEN-VALUES and EIGEN-VECTORS

Let's review some issues we have with matrix multiplication.

First, it doesn't commute: there are many matrices A, B with $AB \neq BA$; e.g., if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ then}$$

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = BA.$$

Second, for A a square matrix, recall (Theorem 1.29) that the solution of the

Difference Equation 1.24

$$\vec{x}_{k+1} = A\vec{x}_k \qquad k = 0, 1, 2, \ldots$$

is

$$\vec{x}_n = A^n \vec{x}_0 \qquad n = 1, 2, 3, \ldots$$

For this and other (see Appendix II) reasons, we need a closed form for powers of A, that is, an explicit formula for A^n that contains only numbers and n.

For example, if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$ then

$$A^2 = A \cdot A = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix},$$

$$A^3 = A \cdot A^2 = \begin{bmatrix} 37 & 54 \\ 81 & 118 \end{bmatrix},$$

. . .

if, in solving the Difference Equation

$$\vec{x}_{k+1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x}_k ,$$

I wanted $A^{1,000,000}$,

I would have to perform 1,000,000 matrix products, unless I can see some pattern appearing in $A, A^2, A^3, \ldots$

Third, and most general and ambiguous, matrix multiplication is hard to understand. Even for a (2×2) matrix A, what A does to vectors in $\mathbb{R}^2$ — that is, the relationship between $\vec{x}$ and $A\vec{x}$ in $\mathbb{R}^2$ — is a weird mixture of rotation and magnification.

Consider now $A \equiv \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. When we multiply

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 3x_2 \end{bmatrix},$$

we understand what is happening to vectors in $\mathbb{R}^2$: the first component is doubled, the second is tripled.

It is not hard to show that

$$A^2 = \begin{bmatrix} 2^2 & 0 \\ 0 & 3^2 \end{bmatrix},$$

$$A^3 = \begin{bmatrix} 2^3 & 0 \\ 0 & 3^3 \end{bmatrix}, \ldots$$

$$A^k = \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix}, \quad k = 1, 2, 3, \ldots;$$

this is what we meant above by a <u>closed form</u> for A^k. If we wanted $A^{1,000,000}$, we would replace k with 1,000,000:

$$A^{1,000,000} = \begin{bmatrix} 2^{1,000,000} & 0 \\ 0 & 3^{1,000,000} \end{bmatrix},$$

with no intermediate powers of A

Thus we can instantly solve

$$\vec{x}_{k+1} = A\vec{x}_k :$$

$$\vec{x}_n = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \vec{x}_0 , \quad n = 1, 2, 3, \ldots$$

$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ is an example of a

<u>diagonal</u> <u>matrix</u>

(see Definition 3.3 and Proposition 3.13).

The point of the discussion in this chapter so far is that diagonal matrices are great. This chapter may be summarized as an attempt to make any matrix behave like a diagonal matrix.

SECTION VIII A: EIGENVALUES, EIGENVECTORS and EIGENSPACES.

Notice that

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This is the first aspect of a diagonal matrix that we would like to generalize.

DEFINITION 8.1 If

$$A\vec{x} = \lambda\vec{x}$$

where $\vec{x}$ is a nontrivial vector and λ is a real number, then λ is an **eigenvalue** for A and $\vec{x}$ is the corresponding **eigenvector**.

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Examples 8.2

(1) $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector for $A = \begin{bmatrix} 2 & 10 \\ 0 & 7 \end{bmatrix}$, since

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

with corresponding eigenvalue 2.

(2) Is $(2, 10, 0)$ an eigenvector for $\begin{bmatrix} 4 & -2 & 19 \\ 5 & -7 & 10 \\ 0 & 0 & 1 \end{bmatrix}$?

If so, what's the eigenvalue?

SOLUTION: Multiply

$$\begin{bmatrix} 4 & -2 & 19 \\ 5 & -7 & 10 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} -12 \\ -60 \\ 0 \end{bmatrix} = (-6) \begin{bmatrix} 2 \\ 10 \\ 0 \end{bmatrix},$$

so yes, eigenvector, eigenvalue (-6)

(3) SAME as (2), for vector $(0,1,2)$.

SOLUTION: $\begin{bmatrix} 4 & -2 & 19 \\ 5 & -7 & 10 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} 36 \\ 13 \\ 2 \end{bmatrix} \neq \lambda \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \\ 2\lambda \end{bmatrix},$$

for any real λ. NO

DEFINITION 8.3 p. 650

If λ is real and A is a square matrix, the

eigenspace of λ for A is

$$E_\lambda \equiv \{ \vec{x} \mid A\vec{x} = \lambda\vec{x} \}.$$

Here we have collected all eigenvectors corresponding to λ and thrown in $\vec{0}$, to make a vector space.

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Example 8.4

Find the eigenspace of $\lambda = 3$ for $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$

SOLUTION:

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longrightarrow$$

$$\begin{aligned} 2x_1 + 2x_2 &= 3x_1 \\ 2x_1 - x_2 &= 3x_2 \end{aligned} \longrightarrow$$

$$\begin{aligned} -x_1 + 2x_2 &= 0 \\ 2x_1 - 4x_2 &= 0 \end{aligned} \rightarrow \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 + 2R_1} \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\rightarrow \quad x_1 - 2x_2 = 0 \quad \rightarrow$

$x_1 = 2x_2, \quad x_2$ arbitrary;

$$E_3 = \left\{ \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} \middle| \; x_2 \text{ arbitrary} \right\}$$

NOTE that we got

the null space of $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$

$= \mathcal{N}(A - 3I)$

THEOREM 8.5

$$E_\lambda = \mathcal{N}(A - \lambda I),$$

for any real λ, square matrix A.

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Example 8.6

Find the eigenspace of the eigenvalue 2 for

$$\begin{bmatrix} 3 & 2 & 3 \\ 1 & 4 & 3 \\ 2 & 4 & 8 \end{bmatrix}.$$

SOLUTION:

$$E_2 = \mathcal{N}\left(\begin{bmatrix} 3 & 2 & 3 \\ 1 & 4 & 3 \\ 2 & 4 & 8 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \mathcal{N}\left(\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \right);$$

we are solving

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow x_1 + 2x_2 + 3x_3 = 0$$

$\rightarrow x_1 = -2x_2 - 3x_3$, x_2, x_3 arbitrary;

$$\boxed{E_2 = \left\{ (-2x_2 - 3x_3,\ x_2,\ x_3) \;\middle|\; x_2, x_3 \text{ arbitrary} \right\}}$$

Given an eigenvalue, getting eigenvectors or eigenspaces is a familiar activity, namely Gauss-Jordan elimination.

Getting eigenvalues is a bigger deal. Recall that a matrix is <u>singular</u> if its null space is nontrivial. Theorem 8.5 now tells us that λ is an eigenvalue for A if and only if $(A - \lambda I)$ is singular; Theorem 5.12 tells us this is equivalent to

$$\det(A - \lambda I) = 0.$$

This motivates the following definition and theorem.

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DEFINITION 8.7

If A is a square matrix, the **characteristic polynomial** of A is

$$c_A(t) \equiv \det(A - tI).$$

THEOREM 8.8

λ is an eigenvalue of A if and only if $c_A(\lambda) = 0$.

Proof: Theorems 8.5 and 5.12

Examples 8.9

Find all eigenvalues.

(1) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Set

$$0 = C_A(t) \equiv \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \det \begin{bmatrix} (1-t) & 2 \\ 3 & (4-t) \end{bmatrix} = (1-t)(4-t) - 3\cdot 2$$

$$= (t^2 - 5t + 4) - 6 = t^2 - 5t - 2 \rightarrow$$

$$\lambda = \frac{1}{2}\left(5 \pm \sqrt{25 - 4(-2)}\right) \quad \left(\begin{array}{l}\text{quadratic} \\ \text{formula}\end{array}\right)$$

$$= \boxed{\frac{1}{2}\left(5 \pm \sqrt{33}\right)}$$

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(2) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$0 = c_A(t) = \det \begin{bmatrix} (1-t) & 0 & 1 \\ 0 & (1-t) & 1 \\ 1 & 1 & -t \end{bmatrix}$$

$$= (1-t)\det\begin{bmatrix} (1-t) & 1 \\ 1 & -t \end{bmatrix} - 0 \det\begin{bmatrix} 0 & 1 \\ 1 & -t \end{bmatrix}$$

$$+ 1 \det \begin{bmatrix} 0 & (1-t) \\ 1 & 1 \end{bmatrix} =$$

$$(1-t)\left[(1-t)(-t) - 1\right] + \left[0 - (1-t)\right] =$$

$$(1-t)\left[(1-t)(-t) - 1 - 1\right] = (1-t)\left[t^2 - t - 2\right]$$

$$= (1-t)(t-2)(t+1) \rightarrow$$ eigenvalues 1, 2, −1

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The eigenvalue 1 is of particular interest; corresponding eigenvectors are given the following names.

DEFINITION 8.10

If $A\vec{x} = \vec{x}$, then $\vec{x}$ is a **fixed point** or **stable point** or **steady state** or **equilibrium** for A.

Note that $\vec{x}$ is an eigenvector corresponding to the eigenvalue one.

When $\vec{x}_0$ is a fixed point of A, the corresponding Difference Equation

$$\vec{x}_{k+1} = A\vec{x}_k$$

has the constant solution

$$\vec{x}_n = \vec{x}_0 \quad (n = 0, 1, 2, \ldots);$$

you never leave the initial state $\vec{x}_0$.

THEOREM 8.11

Every Markov matrix has a fixed point.

Example 8.12

Recall the Moonorgs of Examples 1.25(5) and 7.22(c), with Markov matrix

$$A = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$$

describing moving on or off the moon.

To get fixed points, that is, eigenvectors for the eigenvalue one, we want

$$\mathcal{N}(A-I) = \mathcal{N}\left(\begin{bmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{bmatrix}\right):$$

$$\left[\begin{array}{cc|c} -0.1 & 0.2 & 0 \\ 0.1 & -0.2 & 0 \end{array}\right] \xrightarrow[10R_2]{-10R_1} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2-R_1} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow x_1 - 2x_2 = 0$$

$\rightarrow x_1 = 2x_2$, x_2 arbitrary;

for any x_2, $\begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix}$ is a fixed point for A.

This means that, if there are twice as many Moonorgs off the moon as there are on the moon, the same will be true forever:

$$\begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix}$$

for any x_2; we have a stable population distribution.

REMARKS 8.13

Much more can be said in Example 8.12. For <u>any</u> initial distribution of Moonorgs

$$\vec{x}_0 \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\vec{x}_n = A^n \vec{x}_0$ converges; that is, gets and stays arbitrarily close, to the fixed point

$$\begin{bmatrix} \frac{2}{3}(x_1 + x_2) \\ \frac{1}{3}(x_1 + x_2) \end{bmatrix},$$

as n gets large.

More generally, if A is a Markov matrix with strictly positive entries, then $A^n \vec{x}_0$ converges to a fixed point, for any $\vec{x}_0$, as n gets large.

As mentioned on page 660, having $\vec{x}_0$ be a fixed point of A leads to a very simple solution of the corresponding Difference Equation, since

$$A^n \vec{x}_0 = \vec{x}_0, \quad \text{for } n = 1, 2, 3, \ldots$$

Having $\vec{x}_0$ be <u>any</u> eigenvector of A is almost as simple.

THEOREM 8.14

If $A\vec{x}_0 = \lambda\vec{x}_0$, then

$$A^n\vec{x}_0 = \lambda^n\vec{x}_0, \quad n = 1, 2, 3, \ldots$$

Proof: $A^2\vec{x}_0 = A(A\vec{x}_0)$
$= A(\lambda\vec{x}_0) = \lambda(A\vec{x}_0) = \lambda(\lambda\vec{x}_0)$
$= \lambda^2\vec{x}_0$;

$A^3\vec{x}_0 = A(A^2\vec{x}_0) = A(\lambda^2\vec{x}_0)$
$= \lambda^2(A\vec{x}_0) = \lambda^2(\lambda\vec{x}_0) = \lambda^3\vec{x}_0$,

etc.

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As a corollary, we have one of two methods for solving Difference Equations. This method requires that the initial data $\vec{x}_0$ be a linear combination of eigenvectors. See 8.20 for the other method.

LOCAL METHOD for solving Difference Equations 8.15

If $A\vec{y}_k = \lambda_k \vec{y}_k$ $(k = 1, 2, \ldots, m)$

and $\vec{x}_0 = \alpha_1 \vec{y}_1 + \alpha_2 \vec{y}_2 + \ldots + \alpha_m \vec{y}_m$

(λs and αs are numbers),
then, for $n = 0, 1, 2, \ldots$

$$A^n \vec{x}_0 = \alpha_1 \lambda_1^n \vec{y}_1 + \alpha_2 \lambda_2^n \vec{y}_2 + \cdots + \alpha_m \lambda_m^n \vec{y}_m .$$

Examples 8.16

In each of the following, solve the Difference Equation

$$\vec{x}_{k+1} = A \vec{x}_k \quad (k = 0, 1, 2, \ldots)$$

for the specified A and $\vec{x}_0$.

(a) $A \equiv \begin{bmatrix} 3 & -1 & 1 \\ 1 & 3 & 0 \\ 1 & -1 & 3 \end{bmatrix}$, $\vec{x}_0 \equiv (-1, 1, 2)$.

Multiply: $A\vec{x}_0 = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 3 & 0 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix} = 2\vec{x}_0$, thus

$$\vec{x}_n = A^n \vec{x}_0 = 2^n \vec{x}_0, \quad n = 0, 1, 2, \ldots$$

(b) $A \equiv \begin{bmatrix} -4 & 5 \\ 5 & -4 \end{bmatrix}$, $\vec{x}_0 = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$.

Multiplying $A\vec{x}_0$ shows that $\vec{x}_0$ is not an eigenvector.

To apply 8.15 we need eigenvectors and eigenvalues. Since we have neither, we must begin with eigenvalues:

$$0 = C_A(t) \equiv \det\left(A - t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) =$$

$$\det\begin{bmatrix} (-4-t) & 5 \\ 5 & (-4-t) \end{bmatrix} = (4+t)^2 - 5^2$$

$$= (t^2 + 8t - 9) = (t+9)(t-1)$$

$\rightarrow$ eigenvalues $1, -9$.

Now we get eigenspaces:

$E_1 = \mathcal{N}(A - I) =$

$\mathcal{N}\left(\begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix}\right) \to \left[\begin{array}{cc|c} -5 & 5 & 0 \\ 5 & -5 & 0 \end{array}\right] \to$

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to$

$x_1 - x_2 = 0 \to x_1 = x_2$, x_2 arbitrary;

basis for E_1 is
$\vec{y}_1 = (1,1)$.

$E_{-9} = \mathcal{N}(A + 9I) = \mathcal{N}\left(\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}\right) \to$

$\left[\begin{array}{cc|c} 5 & 5 & 0 \\ 5 & 5 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\to x_1 + x_2 = 0 \to x_1 = -x_2$, x_2 arbitrary;

basis for E_{-9} is $\vec{y}_2 = (1,-1)$

Now we need $\vec{x}_0$ as a linear combination of $\{\vec{y}_1, \vec{y}_2\}$; since this is an orthogonal set, we may use 6.36:

$$\vec{x}_0 = \left(\frac{\vec{x}_0 \cdot \vec{y}_1}{\|\vec{y}_1\|^2}\right)\vec{y}_1 + \left(\frac{\vec{x}_0 \cdot \vec{y}_2}{\|\vec{y}_2\|^2}\right)\vec{y}_2$$

$$= \left(\frac{4}{2}\right)\vec{y}_1 + \left(\frac{-10}{2}\right)\vec{y}_2 = 2\vec{y}_1 - 5\vec{y}_2,$$

so that, by 8.15,

$$\vec{x}_n = A^n\vec{x}_0 = A^n(2\vec{y}_1 - 5\vec{y}_2) =$$

$$2A^n\vec{y}_1 - 5A^n\vec{y}_2 = 2\cdot 1^n\vec{y}_1 - 5(-9)^n\vec{y}_2$$

$$= 2(1,1) - 5(-9)^n(1,-1)$$

$$= (2 - 5(-9)^n,\ 2 + 5(-9)^n)$$

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(c) $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 3 \end{bmatrix}$, $\vec{x}_0 = (-3, -2, 4)$.

USEFUL INFORMATION:

$\{\vec{v}_1 = (-1,-1,1),\ \vec{v}_2 = (-1,1,0),$
$\vec{v}_3 = (-1,0,1)\}$

is a basis of eigenvectors for $\mathbb{R}^3$.

For eigenvalues, multiply:

$A\vec{v}_1 = \vec{v}_1$, $A\vec{v}_2 = 2\vec{v}_2$, $A\vec{v}_3 = 2\vec{v}_3$.

To use 8.15 we again need $\vec{x}_0$ as a linear combination of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. We can't use 6.36, since the basis is not orthogonal!

$$x_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 4 \end{bmatrix} \rightarrow$$

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & -3 \\ -1 & 1 & 0 & -2 \\ 1 & 0 & 1 & 4 \end{array}\right] \rightarrow \cdots$$ (Gauss-Jordan, left to you)

$x_1 = 1, \; x_2 = (-1), \; x_3 = 3$:

$\vec{x}_0 = \vec{v}_1 - \vec{v}_2 + 3\vec{v}_3 \longrightarrow$

$$\vec{x}_n = A^n \vec{x}_0 = A^n \vec{v}_1 - A^n \vec{v}_2 + 3A^n \vec{v}_3$$

$$= 1^n \vec{v}_1 - (2^n)\vec{v}_2 + 3\cdot(2^n)\vec{v}_3$$

$$= (-1,-1,1) - (2^n)(-1,1,0) + 3\cdot(2^n)(-1,0,1)$$

$$= \big((-1 + 2^n - 3\cdot(2^n)),\ (-1 - (2^n)),\ (1 + 3(2^n))\big)$$

(d) A is a (5×5) matrix

such that

$$A\begin{bmatrix}1\\0\\1\\2\\0\end{bmatrix} = \begin{bmatrix}7\\0\\7\\14\\0\end{bmatrix}, \quad A\begin{bmatrix}1\\0\\1\\2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix},$$

$$A\begin{bmatrix}1\\-1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\1\\-1\\0\\0\end{bmatrix}, \quad \vec{x}_0 = \left(\begin{bmatrix}1\\0\\1\\2\\0\end{bmatrix} - \begin{bmatrix}1\\0\\1\\2\\1\end{bmatrix} + 2\begin{bmatrix}1\\-1\\1\\0\\0\end{bmatrix}\right).$$

The hard work has been done

for us.

$$\vec{x}_n = A^n \vec{x}_0 = A^n\begin{bmatrix}1\\0\\1\\2\\0\end{bmatrix} - A^n\begin{bmatrix}1\\0\\1\\2\\1\end{bmatrix} + 2A^n\begin{bmatrix}1\\-1\\1\\0\\0\end{bmatrix}$$

$$= 7^n\begin{bmatrix}1\\0\\1\\2\\0\end{bmatrix} - 0^n\begin{bmatrix}1\\0\\1\\2\\1\end{bmatrix} + 2(-1)^n\begin{bmatrix}1\\-1\\1\\0\\0\end{bmatrix}, \text{ or}$$

$$\vec{x}_n = \begin{bmatrix} 7^n + 2(-1)^n \\ 2(-1)^{n+1} \\ 7^n + 2(-1)^n \\ 2 \cdot 7^n \\ 0 \end{bmatrix}$$

Example 8.17 Use 8.15 to get a closed form for the Fibonacci numbers of Definition 7.25; see Example 7.24.

In the last pages of Chapter VII we have written the

Fibonacci numbers $\{F_N\}_{N=1}^{\infty}$

as coming from the solution

of the Difference Equation

$$\vec{x}_{k+1} = A\vec{x}_k \quad (k=0,1,2,\dots),$$

$$\vec{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{where}$$

$$A \equiv \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \vec{x}_k \equiv \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix},$$

so that

$$\begin{bmatrix} F_N \\ F_{N+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^N \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Our path is clear. We need a basis for $\mathbb{R}^2$ consisting of eigenvectors for $A \equiv \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, and we need to write $(0,1)$ as a linear combination of those eigenvectors.

Eigenvalues 1st:

$$0 = c_A(t) = \det \begin{bmatrix} (-t) & 1 \\ 1 & (1-t) \end{bmatrix}$$

$$= (-t)(1-t) - 1 = t^2 - t - 1 \rightarrow$$

$$t = \tfrac{1}{2}(1 \pm \sqrt{5}) \quad \left(\text{quadratic formula}\right)$$

Eigenvalues are

$\frac{1}{2}(1+\sqrt{5})$ & $\frac{1}{2}(1-\sqrt{5})$.

We need eigenspaces:

$$E_{\frac{1}{2}(1+\sqrt{5})} = \mathcal{N}\left(A - \tfrac{1}{2}(1+\sqrt{5})I_2\right) \to$$

$$\left[\begin{array}{cc|c} -\frac{1}{2}(1+\sqrt{5}) & 1 & 0 \\ 1 & 1-\frac{1}{2}(1+\sqrt{5}) & 0 \end{array}\right] =$$

$$\left[\begin{array}{cc|c} -\frac{1}{2}(1+\sqrt{5}) & 1 & 0 \\ 1 & \frac{1}{2}(1-\sqrt{5}) & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2}(1-\sqrt{5}) & 0 \\ -\frac{1}{2}(1+\sqrt{5}) & 1 & 0 \end{array}\right] \xrightarrow{R_2 + \frac{1}{2}(1+\sqrt{5})R_1}$$

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2}(1-\sqrt{5}) & 0 \\ 0 & 0 & 0 \end{array}\right] \to x_1 + \tfrac{1}{2}(1-\sqrt{5})x_2 = 0$$

$$\boxed{\begin{array}{l}\text{Basis for } E_{\frac{1}{2}(1+\sqrt{5})}: \\ \vec{y}_1 \equiv \begin{bmatrix} (1-\sqrt{5}) \\ -2 \end{bmatrix} \\ E_{\frac{1}{2}(1-\sqrt{5})} \text{ similarly has basis} \\ \vec{y}_2 \equiv \begin{bmatrix} (1+\sqrt{5}) \\ -2 \end{bmatrix}.\end{array}}$$

Since $\{\vec{y}_1, \vec{y}_2\}$ is orthogonal, we may write

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left(\frac{(0,1)\cdot\vec{y}_1}{\|\vec{y}_1\|^2}\right)\vec{y}_1 + \left(\frac{(0,1)\cdot\vec{y}_2}{\|\vec{y}_2\|^2}\right)\vec{y}_2$$

$$= \left(\frac{-2}{(1-\sqrt{5})^2+4}\right)\vec{y}_1 + \left(\frac{-2}{(1+\sqrt{5})^2+4}\right)\vec{y}_2$$

$$= \left(\frac{-2}{1-2\sqrt{5}+5+4}\right)\vec{y}_1 + \left(\frac{-2}{1+2\sqrt{5}+5+4}\right)\vec{y}_2$$

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$$= \left(\frac{-1}{5-\sqrt{5}}\right)\vec{y}_1 + \left(\frac{-1}{5+\sqrt{5}}\right)\vec{y}_2 \longrightarrow$$

$$\begin{bmatrix} F_N \\ F_{N+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^N \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$\left(\frac{-1}{5-\sqrt{5}}\right)\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^N \vec{y}_1 + \left(\frac{-1}{5+\sqrt{5}}\right)\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^N \vec{y}_2 =$$

$$\left(\frac{-1}{5-\sqrt{5}}\right)\left(\tfrac{1}{2}(1+\sqrt{5})\right)^N \vec{y}_1 + \left(\frac{-1}{5+\sqrt{5}}\right)\left(\tfrac{1}{2}(1-\sqrt{5})\right)^N \vec{y}_2 =$$

$$\begin{bmatrix} \left(\frac{-1}{5-\sqrt{5}}\right)\left(\tfrac{1}{2}(1+\sqrt{5})\right)^N(1-\sqrt{5}) + \left(\frac{-1}{5+\sqrt{5}}\right)\left(\tfrac{1}{2}(1-\sqrt{5})\right)^N(1+\sqrt{5}) \\ \left(\frac{-1}{5-\sqrt{5}}\right)\left(\tfrac{1}{2}(1+\sqrt{5})\right)^N(-2) + \left(\frac{-1}{5+\sqrt{5}}\right)\left(\tfrac{1}{2}(1-\sqrt{5})\right)^N(-2) \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{\sqrt{5}}\left(\tfrac{1}{2}(1+\sqrt{5})\right)^N - \frac{1}{\sqrt{5}}\left(\tfrac{1}{2}(1-\sqrt{5})\right)^N \\ \left(\frac{2(5+\sqrt{5})}{25-5}\right)\left(\tfrac{1}{2}(1+\sqrt{5})\right)^N + \left(\frac{2(5-\sqrt{5})}{25-5}\right)\left(\tfrac{1}{2}(1-\sqrt{5})\right)^N \end{bmatrix},$$

so that

$$\boxed{(8.17\ast) \quad F_N = \frac{1}{\sqrt{5}}\left[\left(\tfrac{1}{2}(1+\sqrt{5})\right)^N - \left(\tfrac{1}{2}(1-\sqrt{5})\right)^N\right]}$$

It is hard to believe that (8.17*) is giving us integers, much less the integers defined recursively to be the Fibonacci numbers.

(8.17*), combined with the fact that $\left| \frac{1}{\sqrt{5}} \left(\frac{1}{2}(1-\sqrt{5}) \right)^N \right| < 0.5$, $N = 1, 2, \ldots$, give us the following method for getting F_N quickly:

F_N = the integer closest to $\frac{1}{\sqrt{5}} \left(\frac{1}{2}(1+\sqrt{5}) \right)^N$ (8.17**)

For example,

$$\frac{1}{\sqrt{5}}\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^{10} = 55.00363612\ldots,$$

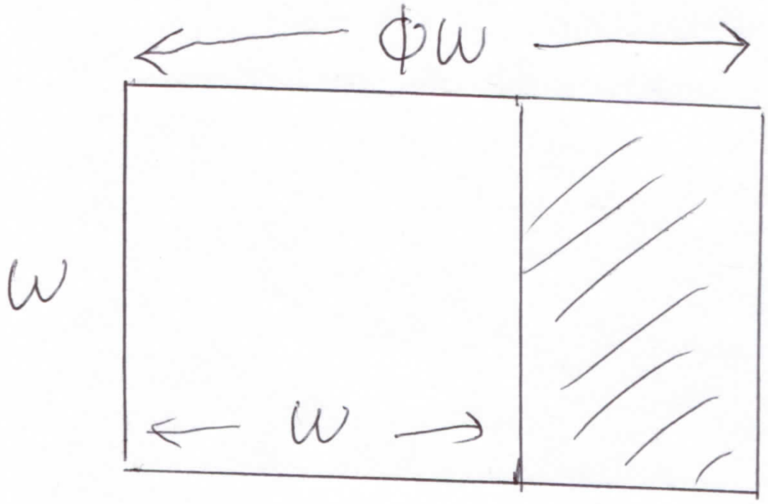
so $F_{10} = 55$.

$$\frac{1}{\sqrt{5}}\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^{21} = 10945.99998\ldots,$$

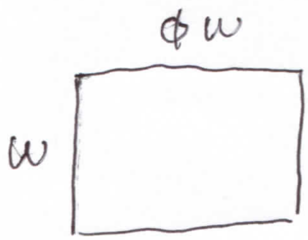
so $F_{21} = 10{,}946$.

$\phi \equiv \frac{1}{2}(1+\sqrt{5})$ is called the **golden ratio**; if a square is removed from one end of a rectangle whose length is ϕ times its width, the rectangle that remains will have the same ratio of length to width.

p. 684



Shaded rectangle has same proportions (length ÷ width) as largest drawn rectangle.



is a **golden rectangle.**