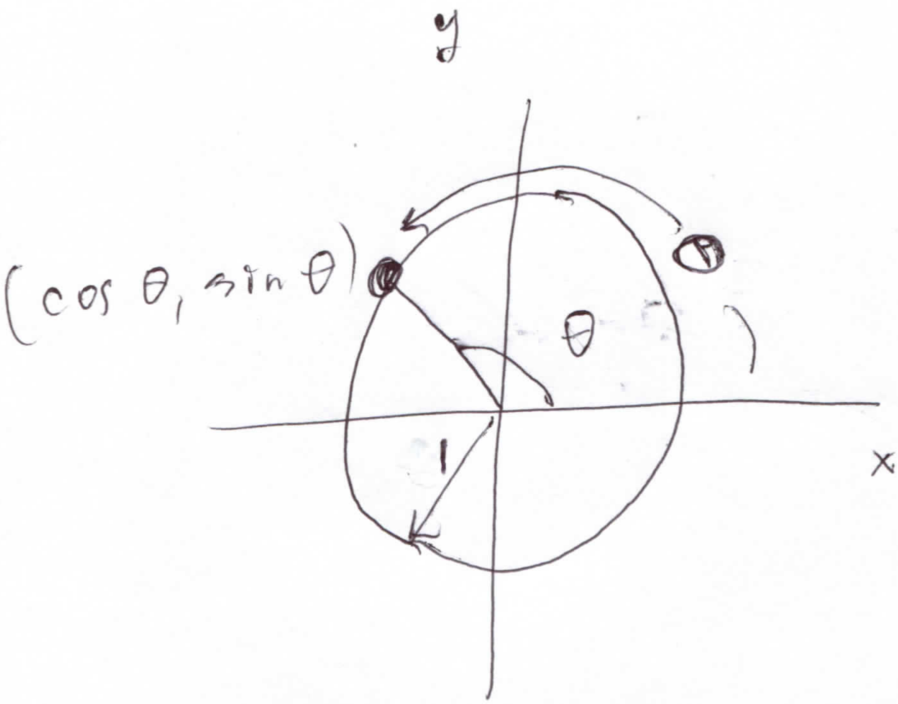
APPENDIX ONE: ROTATION MATRICES

We have constructed standard matrices for rotation only for 45 degrees (7.12) and 90 degrees (Examples 7.11 (2)). To discuss rotations for arbitrary angles, we need some trigonometry; see, e.g., "Trig to the Point,"

www.teacherscholarinstitute.com/FreeMathBooksHighschool.html

All you need is the following picture.



where "cos" is short for "cosine," "sin" is short for "sine," θ is measured in radians as the indicated arclength on the unit circle $x^2+y^2=1$.

θ radians equals $\left(\frac{180\theta}{\pi}\right)$ degrees.

Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

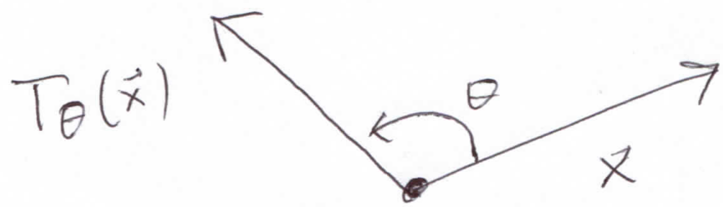
where $i^2 = (-1)$ and "e" is a particular real number, leads quickly to sum of angles formulas

APP 1.1

$$\cos(\theta+\psi) = \cos\theta\cos\psi - \sin\theta\sin\psi$$

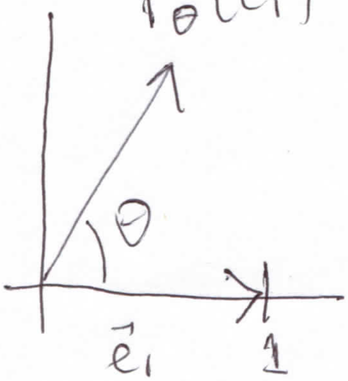
$$\sin(\theta+\psi) = \cos\theta\sin\psi + \cos\psi\sin\theta$$

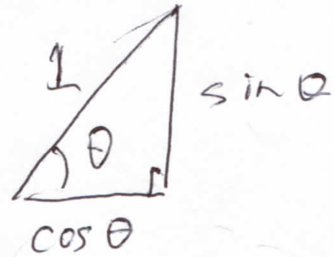
For θ an angle between 0 and 360 degrees, let T_θ be the function that rotates a vector in $\mathbb{R}^2$ θ degrees counterclockwise



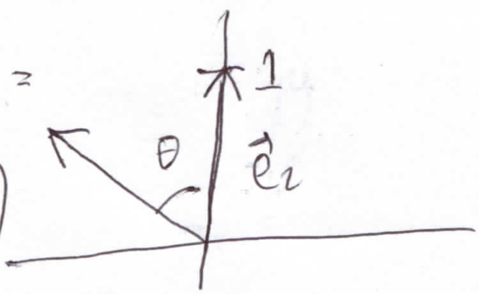
As with 7.12, get the standard matrix T_θ <u>would</u> have if it were linear:

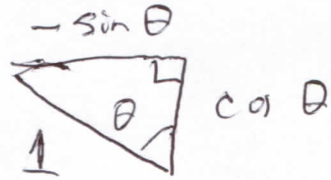
$T_\theta(\vec{e}_1) = (\cos\theta, \sin\theta)$





$T_\theta(\vec{e}_2) = \begin{pmatrix} -\sin\theta, \\ \cos\theta \end{pmatrix}$





→ standard matrix

$$R_\theta = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Again as with 7.12, verify that

$$T_\theta(\vec{x}) = R_\theta \vec{x}$$

for any $\vec{x}$ in $\mathbb{R}^2$:

write nontrivial $\vec{x}$ as $(r\cos\psi, r\sin\psi)$, $r \equiv \|\vec{x}\|$, then

$$R_\theta \vec{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\psi \\ r\sin\psi \end{bmatrix} =$$

$$\begin{bmatrix} r\cos\theta\cos\psi - r\sin\theta\sin\psi \\ r\sin\theta\cos\psi + r\cos\theta\sin\psi \end{bmatrix} =$$

$\begin{bmatrix} r\cos(\theta+\psi) \\ r\sin(\theta+\psi) \end{bmatrix}$, a rotation by θ degrees counterclockwise,

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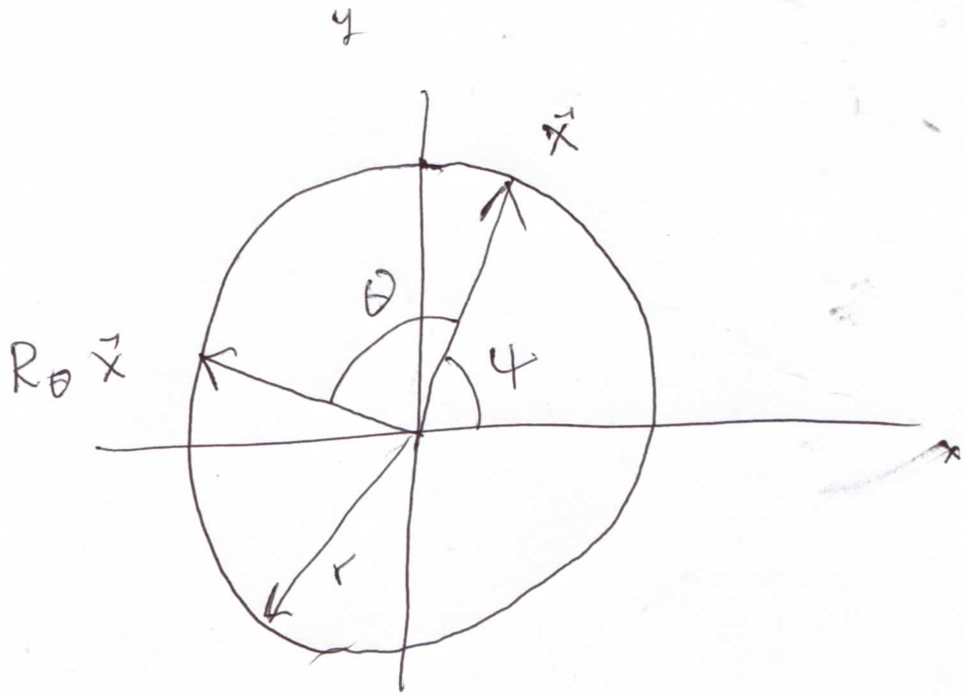

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APPENDIX TWO: SYSTEMS OF DIFFERENTIAL EQUATIONS

This section assumes familiarity with differentiation and exponential functions, showing another application of diagonalizing. The reader should compare this to our use of diagonalizing to solve Difference Equations, as in 8.15 and 8.20.

We address a **system of constant-coefficient differential equations**

$$\frac{dv_1}{dt} = a_{11} v_1(t) + a_{12} v_2(t) + \dots + a_{1m} v_m(t)$$

$$\frac{dv_2}{dt} = a_{21} v_1(t) + a_{22} v_2(t) + \dots + a_{2m} v_m(t) \qquad \text{(APP2.1)}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\frac{dv_m}{dt} = a_{m1} v_1(t) + a_{m2} v_2(t) + \dots + a_{mm} v_m(t)$$

Letting $A \equiv \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}$

$$\vec{v}(t) \equiv \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_m(t) \end{bmatrix},$$

(APP2.1) becomes a single vector-valued differential equation

$$\text{(APP2.2)} \qquad \frac{d\vec{v}}{dt} = A(\vec{v}(t)) .$$

Compare this to (2.5) and (2.15).

In one dimension,

$$\frac{dv}{dt} = av(t) \qquad (a = \text{number})$$

has the solution

$$v(t) = e^{ta}(v(0)),$$

For B a square matrix, define (consistent with B equal to a number)

$$e^{B} \equiv \sum_{k=0}^{\infty} \frac{1}{k!} B^{k} \equiv$$

$$I + B + \frac{1}{2} B^{2} + \frac{1}{6} B^{3} + \ldots$$

THEOREM APP2.3

The solution of (APP2.2) is

$$\vec{v}(t) = \left(e^{tA}\right)\left(\vec{v}(0)\right).$$

THEOREM APP2.4

If $B = P \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_m \end{bmatrix} P^{-1}$,

then

$$e^B = P \begin{bmatrix} e^{\lambda_1} & & & 0 \\ & e^{\lambda_2} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_m} \end{bmatrix} P^{-1}.$$

COROLLARY APP2.5

If $A = P \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_m \end{bmatrix} P^{-1}$, then

the solution of (APP2.2) is

$$\vec{v}(t) = P \begin{bmatrix} e^{t\lambda_1} & & & 0 \\ & e^{t\lambda_2} & & \\ & & \ddots & \\ 0 & & & e^{t\lambda_m} \end{bmatrix} P^{-1}\left(\vec{v}(0)\right)$$

Example APP2.6

Solve

$$\frac{dv_1}{dt} = v_1(t) - v_2(t) - v_3(t)$$

$$\frac{dv_2}{dt} = -v_1(t) + v_2(t) - v_3(t)$$

$$\frac{dv_3}{dt} = v_1(t) + v_2(t) + 3v_3(t)$$

$v_1(0) = 1, \quad v_2(0) = 0, \quad v_3(0) = 2$

SOLUTION: $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 3 \end{bmatrix},$

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \longrightarrow$

our problem is

$$\frac{d\vec{v}}{dt} = A(\vec{v}(t)), \quad \vec{v}(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

We have diagonalized A in Example 8.27:

$$A = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} P^{-1},$$

where $P \equiv \begin{bmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$

$$P^{-1} = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

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By Theorem APP 2.4,

$$e^{tA} = P\begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} P^{-1} = \cdots$$

(left to the reader)

$$\begin{bmatrix} e^{t} & (e^{t}-e^{2t}) & (e^{t}-e^{2t}) \\ (e^{t}-e^{2t}) & e^{t} & (e^{t}-e^{2t}) \\ (-e^{t}+e^{2t}) & (-e^{t}+e^{2t}) & (-e^{t}+2e^{2t}) \end{bmatrix}$$

By Corollary APP 2.5, our solution is

$$\vec{v}(t) = e^{tA}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \cdots \text{ (next page)}$$

$$\begin{bmatrix} 3e^{t} - 2e^{2t} \\ 3e^{t} - 3e^{2t} \\ -3e^{t} + 5e^{2t} \end{bmatrix}$$

REMARK APP2.7

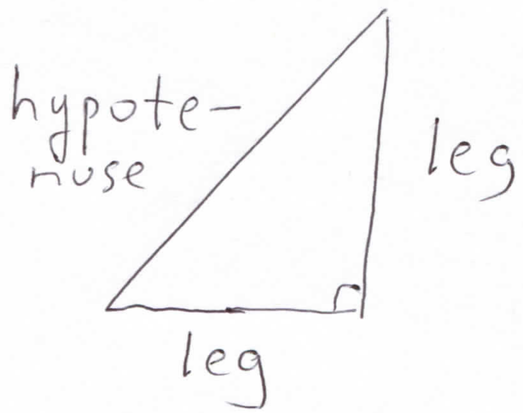
The analogue of 8.15 for (APP2.2) is valid here:

If $A(\vec{v}(0)) = \lambda(\vec{v}(0))$, then

$$\vec{v}(t) = e^{tA}(\vec{v}(0)) = e^{\lambda t}(\vec{v}(0))$$

APPENDIX THREE: PYTHA-GOREAN THEOREM

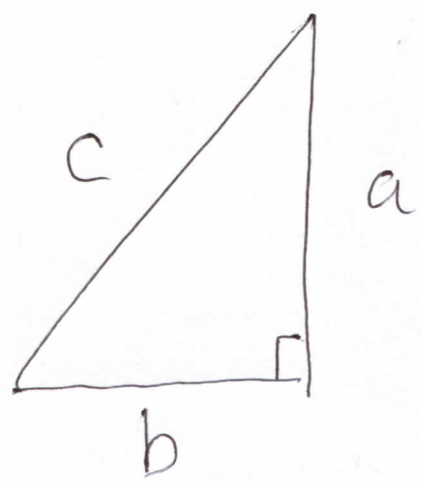
The side opposite the right angle of a right triangle is the **hypotenuse**; the other sides are **legs**.



Let $c \equiv$ length of hypotenuse

$a \equiv$ length of leg

$b \equiv$ length of other leg



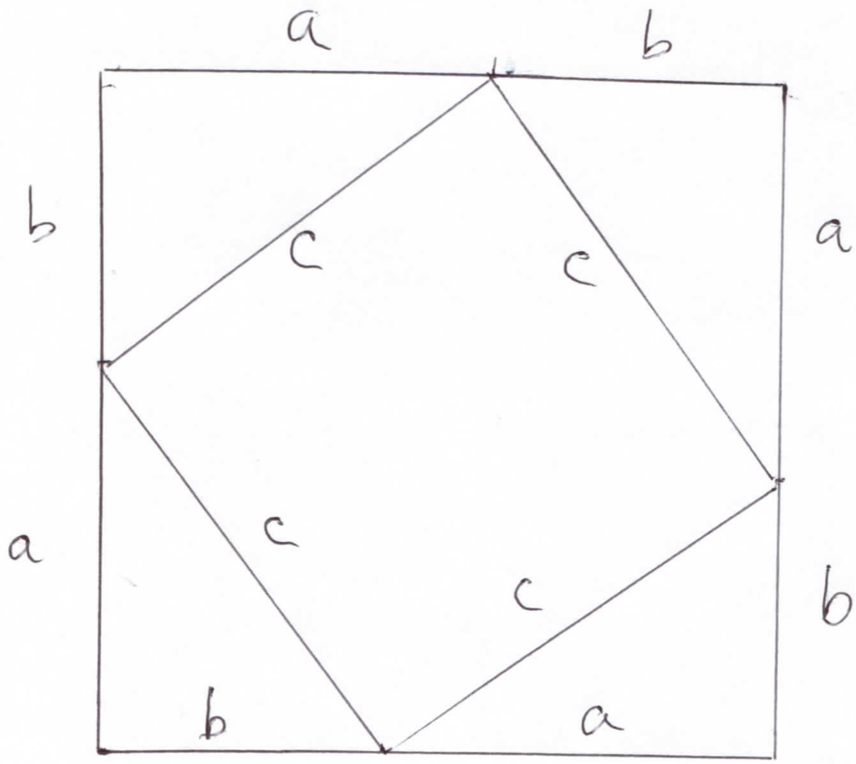
(*)

The only geometry formula we need is

area of square of side w $= w^2$

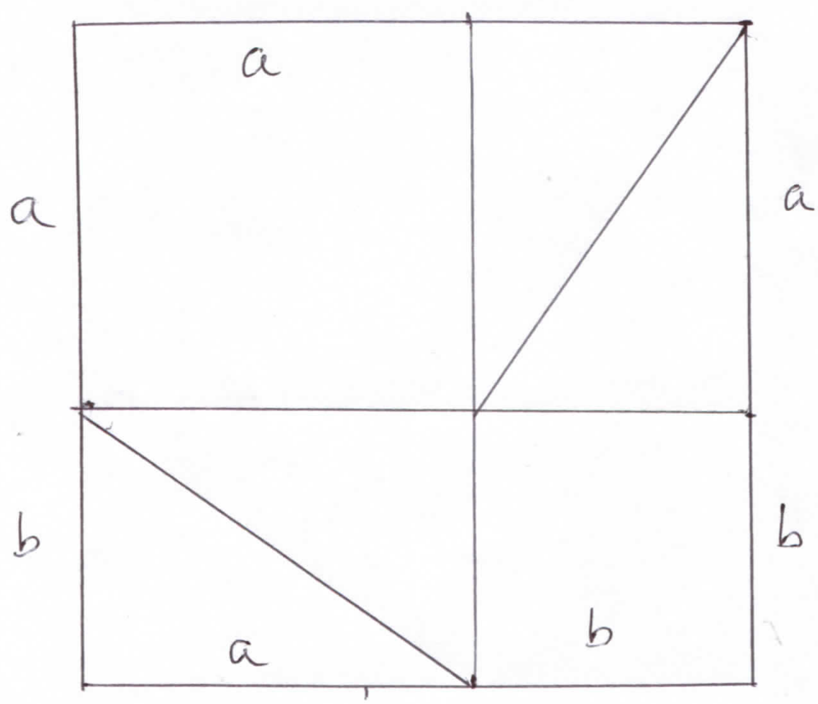
area $= w^2$

Draw two squares of side $(a+b)$, & let $A \equiv$ area of triangle in $(*)$:



$(a+b)^2 =$

$c^2 + 4A$



$(a+b)^2 =$

$a^2 + b^2 + 4A$

Setting the two expressions for $(a+b)^2$ equal gives

$$a^2 + b^2 + 4A = c^2 + 4A$$ or

$$\boxed{\begin{array}{c} a^2 + b^2 = c^2 \\ \text{(Pythagorean theorem)} \end{array}}$$

REMARK The second square of side $(a+b)$ gives us

$$(a+b)^2 = a^2 + 2ab + b^2.$$

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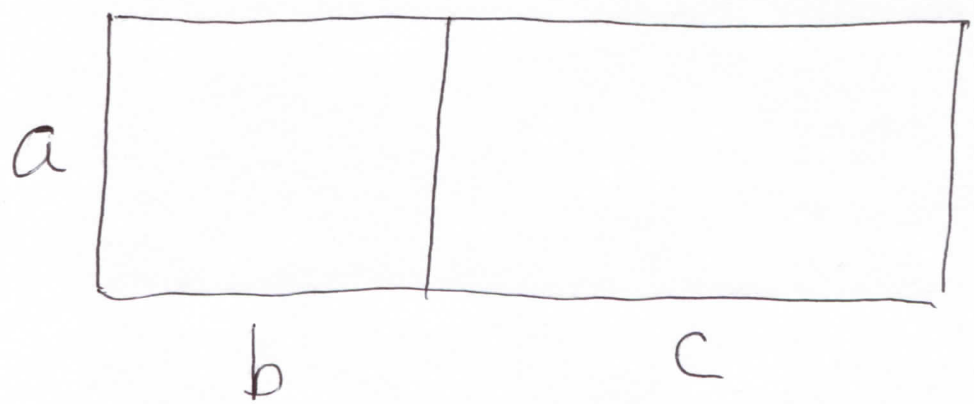
More generally, the distributive law follows geometrically:



$a(b+c)$ = area of biggest rectangle = sum of areas of two smaller rectangles $= ab + ac$.

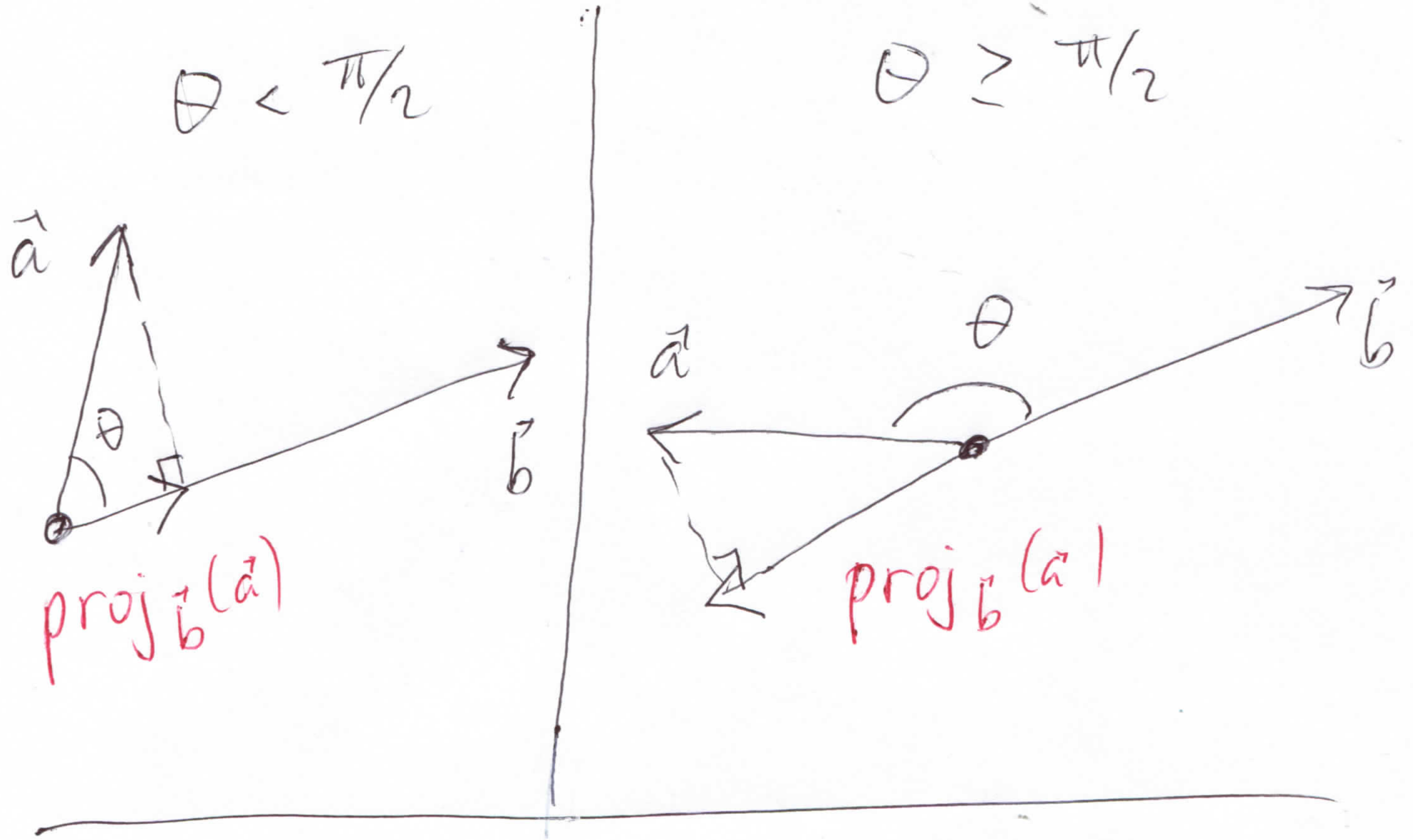
APPENDIX FOUR: ANGLES BETWEEN VECTORS

Definition 6.10 may be considered a dot product characterization of two vectors having an angle measuring 90 degrees between them. As promised in Remarks 6.27, the Cauchy-Schwarz inequality 6.26 will allow us to make similar characterizations of <u>any</u> angle between vectors. As in Appendix One, we'll need some trigonometry:

see in particular the "unit circle" picture on page 749.

Literally in $\mathbb{R}^2$ or figuratively in $\mathbb{R}^n$, $n = 2, 3, 4, \ldots$, for $\vec{a}$, $\vec{b}$ nontrivial vectors, consider the angle of smaller measure θ between $\vec{a}$ and $\vec{b}$, as drawn "below," that is, on the next page, of $\vec{a}$, $\vec{b}$, and $\text{proj}_{\vec{b}}(\vec{a})$, the projection of $\vec{a}$ onto $\vec{b}$.

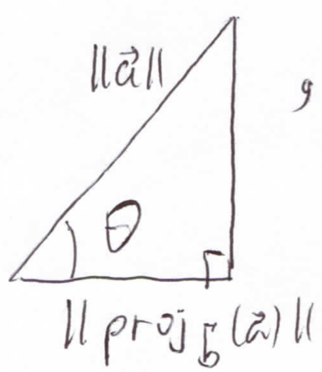
(APP4.1)



Recall that

$$\text{proj}_{\vec{b}}(\vec{a}) = \left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}\right)\vec{b}$$

For $\theta < \frac{\pi}{2}$, since $\text{proj}_{\vec{b}}(\vec{a})$ is a positive multiple of $\vec{b}$, it follows from our formula for $\text{proj}_{\vec{b}}(\vec{a})$ that $(\vec{a} \cdot \vec{b}) > 0$, thus, focussing on the right triangle



, we have

$$\cos\theta = \frac{\|\text{proj}_{\vec{b}}(\vec{a})\|}{\|\vec{a}\|} = \frac{\left\| \left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2} \right) \vec{b} \right\|}{\|\vec{a}\|}$$

$$= \frac{(\vec{a} \cdot \vec{b})}{\|\vec{a}\| \|\vec{b}\|}$$

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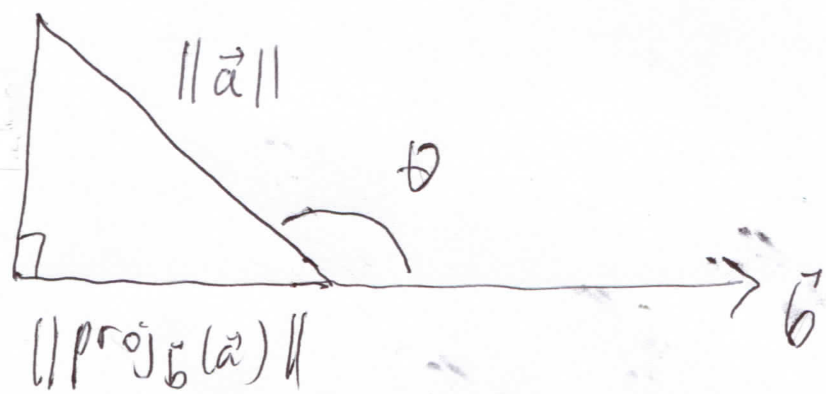
For $\theta \geq \frac{\pi}{2}$, since $\text{proj}_{\vec{b}}(\vec{a})$ is a nonpositive multiple of $\vec{b}$, we now have $(\vec{a} \cdot \vec{b}) \leq 0$, so that, focussing on



,

we now have

$$\cos\theta = -\cos(\pi - \theta) =$$

$$-\frac{\|\text{proj}_{\vec{b}}(\vec{a})\|}{\|\vec{a}\|} = -\left\| \left[\frac{\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}\right)\vec{b}}{\|\vec{a}\|} \right] \right\|$$

$$= \frac{(\vec{a} \circ \vec{b})}{\|\vec{a}\| \|\vec{b}\|};$$

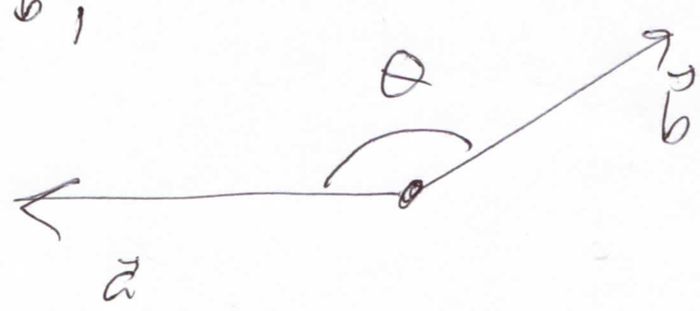
in either case, we have

$$(\text{APP4.2}) \quad \cos\theta = \frac{(\vec{a} \circ \vec{b})}{\|\vec{a}\| \|\vec{b}\|}$$

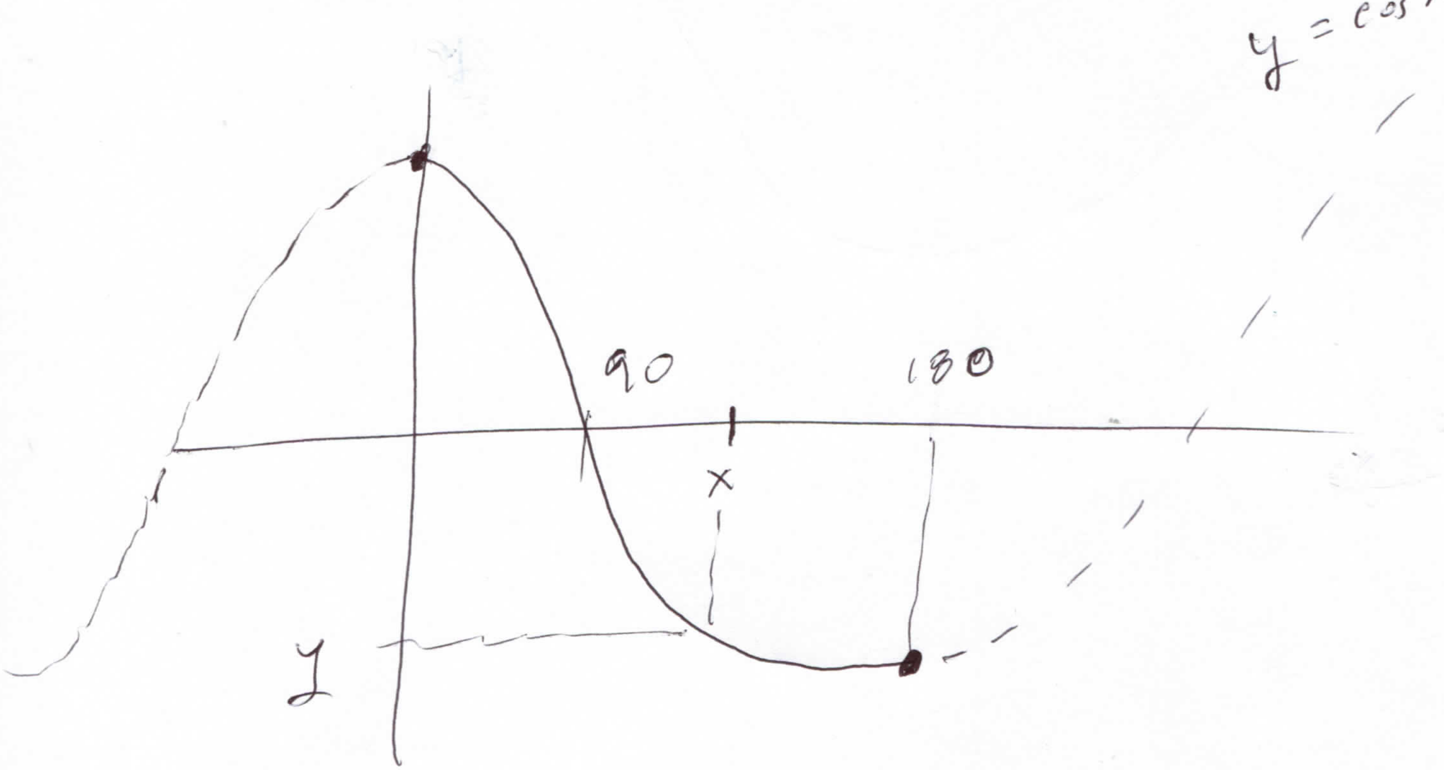
or

$$(\text{APP4.3}) \quad \theta = \cos^{-1}\left(\frac{\vec{a} \circ \vec{b}}{\|\vec{a}\| \|\vec{b}\|}\right),$$

where θ is the measure of the angle of smaller measure between $\vec{a}$ and $\vec{b}$,



and $\cos^{-1}$ is the inverse function of cosine restricted to angles between 0 and 180 degrees.



$\cos^{-1} y = x$ if

$y = \cos x, \quad 0 \le x \le 180$ (degrees)

REMARKS APP4.4

Notice that Cauchy-Schwarz is necessary for (APP4.3) to make sense; $\cos^{-1} y$ is defined only for $|y| \leq 1$, reflecting the fact that $|\cos x| \leq 1$, for all x.

(APP4.3) is a formula wherever angle makes sense, arguably only in $\mathbb{R}^2$; elsewhere it should be taken as a <u>definition</u> of angle measure.