Concise Lecture Notes Series

7

TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI

Review to Prepare for

Pre-Calculus

Dr. Ralph deLaubenfels

TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI TSI

Teacher-Scholar Institute LLC

Columbus, Ohio

2014

ALGEBRA YOU SHOULD KNOW

1.1. set terminology;

1.2. equation terminolgy;

1.3. distributive law;

1.4. natural numbers, integers, rational numbers, irrational numbers, and **real numbers**;

1.5. intervals;

1.6. midpoints;

1.7 absolute value;

1.8. distance between two real numbers;

1.9. how to solve **linear equations and inequalities**;

1.10. how to solve inequalities with absolute values in them;

1.11. factoring and completing the square, for **quadratic expressions**;

1.12. how to recognize and factor **perfect squares, perfect cubes** and **differences of two squares**;

1.13. how to use factorization to solve equations;

1.14. how to use completing the square to solve quadratic equations;

1.15. quadratic formula;

1.16. long division of polynomials (and using to simpify rational functions);

1.17. how to solve **rational inequalities**;

1.18 laws of exponents;

1.19. the **Cartesian plane**;

1.20. distance in the Cartesian plane;

1.21. graph of a circle;

1.22. distance, speed and time.

REVIEW OF ALGEBRA

1. The **set** $\{x \mid ...\}$ reads "The set of all x such that ....."

2. The **equation** $a = b$ reads "a equals b." You must strive to avoid making false statements about equality. If you write "$2 = 3$," for example, there will be wailing and gnashing of teeth.

3. The **distributive law** states that

$$a(b + c) = ab + ac = (b + c)a.$$

Example 4.

$$2(3 + 4) = 2 \cdot 3 + 2 \cdot 4 = 6 + 8 = 14.$$

Or you could have calculated $2(3+4) = 2 \cdot 7 = 14$. When we replace numbers with variables, the distributive law will be unavoidable.

Note, however, that

$$2 \cdot 3 + 4 = 6 + 4 = 10.$$

The point of this example, and the distributive law, is that you should worry about parentheses. ∎

5. The first numbers that arise are the **natural numbers** $\{1, 2, 3, ...\}$. When we throw in zero and additive inverses of natural numbers, we get the **integers** $\{..., -2, -1, 0, 1, 2, ...\}$. A **rational number** is a ratio of two integers (except that we don't allow zero in the denominator).

A **real number** I can't give you a good definition of. Often we say it is any number that can be written as a decimal. These are the numbers that you measure explicitly, such as number of inches of length, number of degrees Fahrenheit of temperature, etc. We like to think of a real number as a point on a line. Unless stated otherwise, all numbers will be real. An **irrational number** is a real number that is not rational.

The most famous irrational number is π, the ratio of the circumference of a circle to its diameter. Another easy way to get irrational numbers is by taking square roots: $\sqrt{2}, \sqrt{3}, \sqrt{5}$ are all irrational; $\sqrt{4} = 2$ is rational. The decimal expansion of a real number tells you if it is rational.

Proposition 6.

(1) *If n is a natural number, then $\sqrt{n}$ is rational only if $\sqrt{n}$ is a natural number.*

(2) *A real number is rational if and only if its decimal expansion terminates or repeats.*

Repetition is indicated by a bar. For example, $2.1\overline{34}$ means 2.1343434....

Example 7. a. Write $3.\overline{2}$ as a ratio of integers.
b. Write $.1\overline{03}$ as a ratio of integers.

Solutions. a. Let's give $3.\overline{2}$ a name; x is both popular and traditional. Note that $10x = 32.2222...$, while $x = 3.2222...$, so

$$9x = 10x - x = 29, \quad \text{so} \quad x = \frac{29}{9}.$$

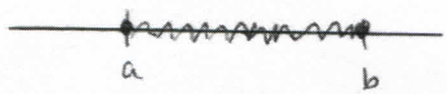
b. Let $x \equiv .1\overline{03} = .1030303...$; then

$$99x = 100x - x = 10.303030... - .1030303... = 10.2, \quad \text{so} \quad x = \frac{10.2}{99} = \frac{102}{990} = \frac{17}{165}.$$

∎

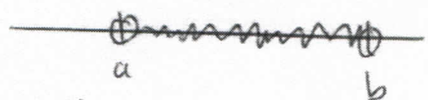
8. A **closed interval** is

$$[a, b] \equiv \{x \,|\, a \leq x \leq b\}.$$



An **open interval** is

$$(a, b) \equiv \{x \,|\, a < x < b\},$$



or

$$(a, \infty) \equiv \{x \,|\, x > a\} \quad \text{or} \quad (-\infty, b) \equiv \{x \,|\, x < b\}.$$

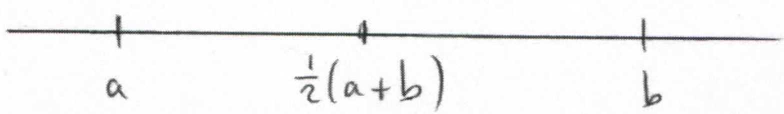
Then there are half-open intervals:

$$(a, b] \equiv \{x \,|\, a < x \leq b\} \quad \text{or} \quad [a, b) \equiv \{x \,|\, a \leq x < b\}$$

or

$$[a, \infty) \equiv \{x \,|\, x \geq a\} \quad \text{or} \quad (-\infty, b] \equiv \{x \,|\, x \leq b\}.$$

9. The **midpoint** of the interval $[a, b]$ is $\frac{1}{2}(a + b)$.

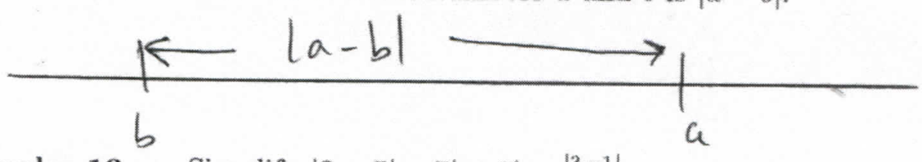


10. The **absolute value** is defined by

$$|x| \equiv \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

For example, $|2| = 2, |0| = 0, |-2| = -(-2) = 2$. Always the absolute value of a real number is nonnegative.

11. The **distance** between two real numbers a and b is $|a - b|$.



Examples 12. a. Simplify $|3 - 5| - 7| - 2| + \frac{|3-1|}{3}$.
b. Find the distance between 19 and -3.

Solutions. a. Work inside the absolute values first:

$$|3 - 5| - 7| - 2| + \frac{|3 - 1|}{3} = | - 2| - 7 \cdot 2 + \frac{|2|}{3} = 2 - 14 + \frac{2}{3} = -\frac{34}{3}.$$

b. $|19-(-3)| = |22| = 22$. NOTE that you'd get the same answer with $|(-3)-19|$. ∎

There are many things you may legally do to both sides of an equation. "Legal" here means that a true statement will not be changed into a false statement. You may add the same number to both sides. You may multiply both sides by the same number. You may raise both sides to any integral power.

Examples 13. Solve each of the following for x.
a. $3x - 5 = \frac{9}{2}x + 7$.
b. $\sqrt{x} + 5 = 7$.
c. $xy + 3t = z$.

Solutions. a. Add 5 to both sides: $3x = \frac{9}{2}x + 12$; subtract $\frac{9}{2}x$ from both sides: $12 = (3 - \frac{9}{2})x = -\frac{3}{2}x$; multiply both sides by $-\frac{2}{3}$, to get $x = 12(-\frac{2}{3}) = -8$.
b. Subtract 5 from both sides: $\sqrt{x} = 2$; now square both sides: $x = (\sqrt{x})^2 = 2^2 = 4$.
c. Subtract $3t$ from both sides: $xy = z - 3t$; now divide both sides by y: $x = \frac{z-3t}{y}$. ∎

14. A **linear expression** has the form $ax + b$, for some numbers a, b, and a variable x. A **linear equation** is one that can be equivalently written in the form $ax + b = 0$. Example 13(a) was an example of a linear equation.

15. Equations could also be called equalities. We also have **inequalities**. The expression "$a < b$" reads "a is less than b." Similarly, "$a \leq b$" reads "a is less than or equal to b."

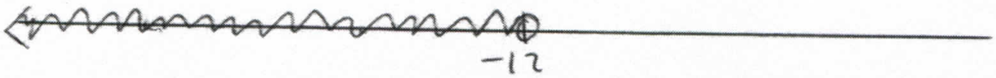
You may solve inequalities in almost the same way you solve equations. You may add the same number to both sides, and you may multiply both sides by the same *positive* number.

Examples 16. Solve each of the following for whatever variable is present.
a. $3x - 5 > 5x + 19$.
b. $\frac{2}{3} + \frac{5y}{6} \leq \frac{7}{12}$.

c. $\frac{z}{4} + \frac{z}{6} + 4 < 3z - 1$.

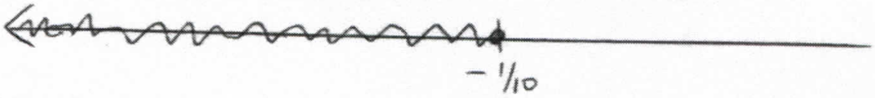
Solutions. a. Subtract $3x$ from both sides: $-5 > 2x + 19$; subtract 19 from both sides: $-24 > 2x$; now divide both sides by 2: $-12 > x$ is the solution.

On a number line, this looks like



b. Let's clear denominators by multiplying both sides by the least common denominator 12: $8 + 10y \leq 7$; subtract 8 from both sides: $10y \leq -1$; now divide both sides by 10: $y \leq -\frac{1}{10}$ is the solution.

This solution is represented on the number line in the following way.



c. Multiply both sides by the least common denominator 12: $3z+2z+48 < 36z-12$, so $5z + 48 < 36z - 12$; subtract $5z$ from, and add 12 to, both sides: $60 < 31z$; now divide both sides by 31: $\frac{60}{31} < z$ (OR, if you prefer, $z > \frac{60}{31}$). ∎

If you multiply both sides of an inequality by a negative number, you must reverse the direction of the inequality. For example, $-x < 5$ becomes $x > -5$ (multiply both sides by -1), as is illustrated by $(-3) > -5$ but $-(-3) = 3 < 5$.

Examples 17. Solve each of the following.

a. $t - \frac{9t}{2} > -2$.

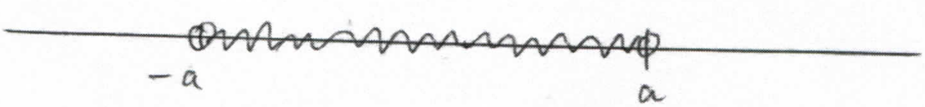
b. $3 - \frac{x}{2} \leq -1$.

Solutions. a. Since $(1 - \frac{9}{2}) = -\frac{7}{2}$, we have $(-\frac{7}{2})t > -2$; multiplying both sides by $-\frac{2}{7}$, this becomes $t < \frac{4}{7}$.

b. Multiplying both sides by -2 gives us $-6 + x \geq 2$, so that $x \geq 8$. ∎

18. Absolute values in inequalities. Suppose $a \geq 0$. For finding solutions, you should rewrite

$$|x| < a \quad \text{as} \quad -a < x < a,$$



and

$$|x| > a \quad \text{as} \quad x > a \text{ or } x < -a.$$

Likewise for $\leq, \geq$.

Examples 19. Solve each of the following and graph your solution.

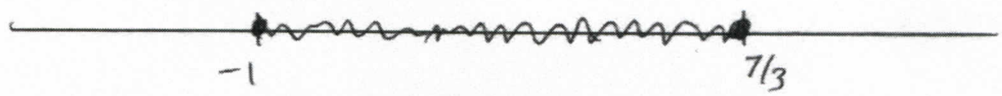
a. $|3p - 2| \leq 5$.

b. $5 - 2|3x + 6| < 1$.

c. $2 \leq -|x - 5|$.

Solutions. a. This is $-5 \leq 3p - 2 \leq 5$; add 2 everywhere: $-3 \leq 3p \leq 7$; now divide by 3: $-1 \leq p \leq \frac{7}{3}$ is the solution.

Here is the graph of the solution.



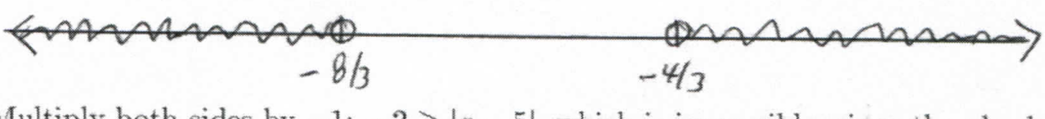
b. We need to isolate the absolute value on one side of the inequality first. Add $2|3x + 6|$ to, and subtract 1 from, both sides: $4 < 2|3x + 6|$; divide both sides by 2: $2 < |3x + 6|$; this translates to

$$3x + 6 > 2 \quad \text{or} \quad 3x + 6 < -2.$$

Solve those two inequalities separately (subtract 6, then divide by 3, on both sides), to get the answer

$$x > -\frac{4}{3} \quad \text{or} \quad x < -\frac{8}{3}.$$

Here is the graph of the solution.



c. Multiply both sides by -1: $-2 \geq |x - 5|$, which is impossible, since the absolute value of any number is ≥ 0.

The answer is "no solution." ∎

20. A **quadratic expression** has the form $ax^2 + bx + c$, for some real numbers a, b, c and a variable x. A **quadratic equation** is one that can be written in the form $ax^2 + bx + c = 0$. To **factor** a quadratic expression means to write it as $a(x + \alpha)(x + \beta)$, a product of linear terms.

For example, $x^2 + 5x + 6 = (x + 3)(x + 2)$. If you were told to "factor $x^2 + 5x + 6$," your answer would be "$(x + 3)(x + 2)$," the *factorization* of $x^2 + 5x + 6$. $(x + 3)$ and $(x + 2)$ are also called *factors* of $x^2 + 5x + 6$.

To see how to factor, think backwards using the distributive law:

$$(x+a)(x+b) = x(x+b)+a(x+b) = x^2+bx+ax+ab = x^2+(a+b)x+ab. \quad (21)$$

21. If a quadratic expression cannot be factored into a product of linear terms with integral coefficients (that is, written as $(ax+b)(cx+d)$, for some integers a, b, c, d), it is said to be **irreducible over the integers**.

Examples 22. For each of the following, factor into a product of linear terms with integral coefficients, or assert that the expression is irreducible over the integers.

a. $x^2 - 3x - 4$.

b. $x^2 - 2x + 6$.

Solutions. a. We want numbers a, b, so that $(x+a)(x+b) = x^2 - 3x - 4$. By (21), $ab = -4$. This means our (integral) possibilities are $a = -1, b = 4$, or $a = 1, b = -4$, or $a = 2, b = -2$. I'll try them out:
$(x-1)(x+4) = x(x+4) - (x+4) = x^2 + 4x - x - 4 = x^2 + 3x - 4$; NOPE.
$(x+1)(x-4) = x(x-4) + (x-4) = x^2 - 4x + x - 4 = x^2 - 3x - 4$; YES.
Our answer is $(x+1)(x-4)$.

b. Now we want a, b so that $(x+a)(x+b) = x^2 - 2x + 6$. By (21), $ab = 6$. Our (integral) possibilities for a, b are $a = 1, b = 6$, or $a = -1, b = -6$, or $a = 2, b = 3$, or $a = -2, b = -3$.

I will try them all.
$(x+1)(x+6) = ... = x^2 + 7x + 6$.
$(x-1)(x-6) = ... = x^2 - 7x + 6$.
$(x+2)(x+3) = ... = x^2 + 5x + 6$.
$(x-2)(x-3) = ... = x^2 - 5x + 6$.

None of the possible integral choices of a and b give us $(x+a)(x+b) = x^2 - 2x + 6$. Thus the answer is "$x^2 - 2x + 6$ is irreducible over the integers." ∎

Remark. You could also choose your a, b, in Example 22a, by checking that $ab = -4$ and $a + b = -3$ (see (21)). Similarly, in Example 22b, you want a, b so that $ab = 6, a + b = -2$.

23. Some factorization rules:

a. $a^2 - b^2 = (a+b)(a-b)$ (Difference of two squares)

b. $a^2 + 2ab + b^2 = (a+b)^2$ (Perfect square)

c. $a^3 + 3a^2b + 3ab^2 + b^3 = (a+b)^3$ (Perfect cube)

d. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ (Difference of two cubes)

Examples 24. Factor each of the following.

a. $36x^2 - 4y^4$.

b. $4x^2 - 12x + 9$.

c. $27x^6 - 8$.

d. $8x^3 - 12x^2 + 6x - 1$.

Solutions. a. This is a difference of two squares, as in 23a, with $a = 6x, b = 2y^2$, so by 23a, our factorization is $(6x + 2y^2)(6x - 2y^2)$.

b. This is a perfect square, as in 23b, with $a = 2x, b = -3$, so our factorization is $(2x - 3)^2$.

c. This is a difference of two cubes, as in 23d, with $a = 3x^2, b = 2$, so our factorization is $(3x^2 - 2)(9x^4 + 6x^2 + 4)$.

d. This is a perfect cube, as in 23c, with $a = 2x, b = -1$, so our factorization is $(2x - 1)^3$. ▮

When the leading coefficient of a quadratic expression is not one, much more trial and error is required to factor it. Think backwards, as in (21), from

$$(ax + b)(cx + d) = (ac)x^2 + (bc + ad)x + (bd). \tag{25}$$

Example 26. Factor $2y^2 - 3y - 2$ into a product of linear terms with integral coefficients.

Solution. We want $(ay + b)(cy + d) = 2y^2 - 3y - 2$; by (25), $ac = 2$ and $bd = -2$. Let's charge through some possibilities.

$a = 1, c = 2, b = 1, d = -2$? $(y+1)(2y-2) = y(2y-2)+(2y-2) = 2y^2-2y+2y-2 = 2y^2 - 2$. NOPE

$a = -1, c = -2, b = 1, d = -2$? $(-y + 1)(-2y - 2) = -y(-2y - 2) + (-2y - 2) = 2y^2 + 2y - 2y - 2 = 2y^2 - 2$. NOPE

$a = 2, c = 1, b = 1, d = -2$? $(2y+1)(y-2) = 2y(y-2)+(y-2) = 2y^2-4y+y-2 = 2y^2 - 3y - 2$. YES

Our answer is $2y^2 - 3y - 2 = (2y + 1)(y - 2)$. ▮

27. How to use factorization to solve equations: After factoring, use the fact that $ab = 0$ if and only if $a = 0$ or $b = 0$.

Examples 28. Solve each of the following.

a. $x^2 + x = 2$.

b. $2y^2 = 1 - y$.

Solutions. Whenever you solve by factoring, first make sure you have zero on one side of the equation.

a. $x^2 + x - 2 = 0$, so we factor $x^2 + x - 2$, that is, write it as $(x + a)(x + b)$. By (21), $ab = -2$, so either $a = 1, b = -2$, or $a = -1, b = 2$.

TRY $(x + 1)(x - 2) = x(x - 2) + (x - 2) = x^2 - 2x + x - 2 = x^2 - x - 2$. NOPE

$(x-1)(x+2) = x(x+2) - (x+2) = x^2 + 2x - x - 2 = x^2 + x - 2$. YES.

So we have $(x-1)(x+2) = 0$, which means that either $(x-1) = 0$ or $(x+2) = 0$. Solving each of these two equations separately gives us two solutions: $x = 1$ or $x = -2$.

b. $2y^2 + y - 1 = 0$. We must factor $2y^2 + y - 1)$ into $(ay+b)(cy+d)$. By (25), $ac = 2, bd = -1$.

TRY $a = 1, c = 2, b = 1, d = -1$: $(y+1)(2y-1) = y(2y-1) + (2y-1) = 2y^2 - y + 2y - 1 = 2y^2 + y - 1$. YES, we're in luck.

So we have $(y+1)(2y-1) = 0$, which means that either $(y+1) = 0$ or $(2y-1) = 0$. We get the two solutions $y = -1$ or $y = \frac{1}{2}$. ∎

29. Completing the square means rewriting a quadratic expression in the form

$$\alpha(x-\beta)^2 + \gamma,$$

for some real numbers α, β, γ.

Note that $(x-\beta)^2 = x^2 - 2\beta x + \beta^2$. Thus, given

$$y = ax^2 + bx + c,$$

to complete the square, you should

(1) subtract c from both sides:

$$y - c = ax^2 + bx,$$

(2) divide both sides by a:

$$\frac{y-c}{a} = x^2 + \frac{b}{a}x,$$

(3) take half of $\frac{b}{a}$, square it, then add that number $\left(\left(\frac{b}{2a}\right)^2\right)$ to both sides:

$$\frac{y-c}{a} + \left(\frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2,$$

(4) rewrite the right hand side as a perfect square:

$$\frac{y-c}{a} + \left(\frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)^2,$$

(5) then solve for y:

$$y = a\left(x + \frac{b}{2a}\right)^2 + \left[c - a\left(\frac{b}{2a}\right)^2\right].$$

Don't memorize this as any kind of formula; rather, perform the routine I just described with whatever specific numbers you have, when you need to complete the square.

Examples 30. In each of the following, complete the square.
a. $x^2 - 12x + 1$.
b. $3x^2 + 6x - 5$.

Solutions. a. Write $y = x^2 - 12x + 1$, then follow steps (1)–(5) in 29:

$$y - 1 = x^2 - 12x \quad \rightarrow \quad y - 1 + (-6)^2 = x^2 - 12x + (-6)^2 = (x-6)^2,$$

so that, solving for y, we get

$$x^2 - 12x + 1 = y = (x-6)^2 - 35.$$

b. Set $y = 3x^2 + 6x - 5$, so that, following steps (1)–(5) in 29,

$$y + 5 = 3x^2 + 6x = 3(x^2 + 2x) \quad \rightarrow \quad \frac{y+5}{3} = x^2 + 2x,$$

and

$$\frac{y+5}{3} + 1^2 = x^2 + 2x + 1^2 = (x+1)^2;$$

now solve for y:

$$\frac{y+5}{3} = (x+1)^2 - 1 \quad \rightarrow \quad y + 5 = 3(x+1)^2 - 3 \quad \rightarrow \quad y = 3(x+1)^2 - 8.$$

You should write your answer as

$$3x^2 + 6x - 5 = 3(x+1)^2 - 8.$$

∎

Examples 31. Solve each of the following by completing the square.
a. $t^2 - 3t + 2 = 0$.
b. $2x^2 + 16x - 1 = 4$.

Solutions. a. First subtract 2 from both sides:

$$-2 = t^2 - 3t.$$

Take half of -3, square it, and add it to both sides:

$$\left(-\frac{3}{2}\right)^2 - 2 = t^2 - 3t + \left(-\frac{3}{2}\right)^2 = \left(t - \frac{3}{2}\right)^2,$$

so that

$$\left(t-\frac{3}{2}\right)=\pm\sqrt{\left(-\frac{3}{2}\right)^2-2}=\pm\sqrt{\frac{1}{4}}=\pm\frac{1}{2},$$

and we get the two solutions $t=\frac{3}{2}\pm\frac{1}{2}$, which simplifies to $t=1$ or 2.
b. Add 1 to both sides, then divide both sides by 2:

$$x^2+8x=\frac{5}{2}.$$

Take half of 8, square it, and add that to both sides:

$$16+\frac{5}{2}=x^2+8x+16=(x+4)^2;$$

thus,

$$(x+4)=\pm\sqrt{16+\frac{5}{2}}=\pm\sqrt{\frac{37}{2}},$$

so that the solutions are $x=-4\pm\sqrt{\frac{37}{2}}$. ∎

If we solve an arbitrary quadratic equation

$$ax^2+bx+c=0 \tag{32}$$

by completing the square: $x^2+\frac{b}{a}x=-\frac{c}{a}$, so that

$$\left(x+\left(\frac{b}{2a}\right)\right)^2=x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}=\frac{b^2-4ac}{4a^2},$$

thus

$$x=-\frac{b}{2a}\pm\sqrt{\frac{b^2-4ac}{4a^2}}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}, \tag{33}$$

we obtain what is called the **quadratic formula**, giving the solutions to (32).

Examples 34. In each of the following, find all real solutions with the quadratic formula.

a. $3z^2-2z-4=0$.

b. $\frac{x^2+1}{2x+1}=x-5$.

c. $x^2+x+1=0$.

Solutions. a. Here $a=3, b=-2, c=-4$, so our solutions are

$$z=\frac{-(-2)\pm\sqrt{(-2)^2-4(3)(-4)}}{2(3)}=\frac{2\pm\sqrt{52}}{6}.$$

b. Getting rid of fractions is often a good strategy. Multiply both sides by $2x+1$, to get

$$x^2+1=(2x+1)(x-5)=2x^2-9x-5,$$

or

$$x^2-9x-6=0.$$

Now use the quadratic formula with $a=1, b=-9, c=-6$, to get

$$x=\frac{-(-9)\pm\sqrt{(-9)^2-4(1)(-6)}}{(2)(1)}=\frac{9\pm\sqrt{105}}{2}.$$

c. Here $a=1=b=c$, so by the quadratic formula,

$$x=\frac{-1\pm\sqrt{1^2-4}}{2}=\frac{-1\pm\sqrt{-3}}{2}.$$

There is no real square root of a negative number. So the correct answer is "no real solutions." ▮

35. The expression b^2-4ac inside the square root in the quadratic formula (33) is called the **discriminant** of the quadratic expression ax^2+bx+c. This tells you the number of real solutions of (32):

1. $b^2-4ac>0$: two real solutions
2. $b^2-4ac=0$: one real solution
3. $b^2-4ac<0$: no real solutions.

Examples 36. Without solving, find how many real solutions each equation has.

a. $2q^2+q-4=0$.

b. $-z^2+2z-100=0$.

c. $4x^2+4x+1=0$.

Solutions. a. Here $a=2, b=1, c=-4$, so the discriminant

$$b^2-4ac=1^2-4(2)(-4)=33>0,$$

thus the equation has two real solutions.

b. Here the discriminant is $2^2-4(-1)(-100)=-396$, so there are no real solutions.

c. Now the discriminant is $4^2-4(4)(1)=0$, so there is exactly one real solution. ▮

37. **A polynomial expression** is

$$a_0+a_1x+a_2x^2+\cdots+a_Nx^N,$$

where $a_0, a_1, ..., a_N$ are numbers; if $a_N\neq 0$, then N is the *degree* of the polynomial.

38. Long division of polynomials. Factor each of the following.

a. $x^3 + 2x^2 - x - 2$.

b. $2y^3 - 3y^2 + 1$.

Solutions. a. We'd like to write $x^3 + 2x^2 - x - 2 = (x+a)(x+b)(x+c)$, with a, b, c integers. This means that $abc = -2$, so $a = \pm 1$ or ± 2. What we do now is use long division to divide $x^3 + 2x^2 - x - 2$, by $(x \pm 1)$ or $(x \pm 2)$. Which of those four linear terms I try is a random guess:

$$x^2 + x - 2$$
$$x+1 \,\overline{)\, x^3 + 2x^2 - x - 2}$$
$$-(x^3 + x^2)$$
$$x^2 - x$$
$$-(x^2 + x)$$
$$-2x - 2$$
$$-(-2x - 2)$$
$$0$$

This tells us that $x^3 + 2x^2 - x - 2 = (x+1)(x^2 + x - 2)$. As with Example 22, $(x^2 + x - 2)$ factors into $(x+2)(x-1)$. So we have

$$x^3 + 2x^2 - x - 2 = (x+1)(x-1)(x+2).$$

b. Here we'd like $2y^3 - 3y^2 + 1 = (ay \pm 1)(by \pm 1)(cy \pm 1)$, for a, b, c integers, so that $abc = 2$. This means that a could be ± 1 or ± 2, and I'll just guess, and try dividing by $(y+1)$.

$$2y^2 - 5y + 5$$
$$y+1 \,\overline{)\, 2y^3 - 3y^2 + 1}$$
$$-(2y^3 + 2y^2)$$
$$-5y^2 + 1$$
$$-(-5y^2 - 5y)$$
$$5y + 1$$
$$-(5y + 5)$$
$$-4$$

Since we get a remainder, we conclude that $(y+1)$ does not divide $2y^3 - 3y^2 + 1$. I'll leave it to you to perform long division of $(y-1)$ into $2y^3 - 3y^2 + 1$; this time you should get

$$2y^3 - 3y^2 + 1 = (y-1)(2y^2 - y - 1).$$

Now we factor $(2y^2 - y - 1)$. We want it equal to $(ay + b)(cy + d)$, where $ac = 2, bd = -1$. I'll start trying:

$(y+1)(2y-1) = \ldots = 2y^2 + y - 1$. NO

$(y-1)(2y+1) = ... = 2y^2 - y - 1$. YES.

Here's our final factorization: $2y^3 - 3y^2 + 1 = (y-1)^2(2y+1)$. ∎

Examples 39. Solve each of the following.

a. $(z-1)(z+9)(z-25) = 0$.

b. $x^3 - 3x^2 + x = 3$.

Solutions. a. Either $(z-1) = 0, (z+9) = 0$ or $(z-25) = 0$. This gives us the three solutions $z = 1, -9$ or 25.

b. Rewrite this as

$$x^3 - 3x^2 + x - 3 = 0.$$

We would like to factor $x^3 - 3x^2 + x - 3$. If we want to write it as $(x+a)(x+b)(x+c)$, then $abc = -3$, so $a = \pm 1$ or ± 3. In fact, if you try to divide $x^3 - 3x^2 + x - 3$ by $(x \pm 1)$ or $(x+3)$, you'll get a remainder. I'll try long division by $(x-3)$.

$$\begin{array}{r} x^2 + 1 \\ x-3 \,\overline{)\, x^3 - 3x^2 + x - 3} \\ -(x^3 - 3x^2) \\ \hline x - 3 \\ -(x-3) \\ \hline 0 \end{array}$$

Thus $x^3 - 3x^2 + x - 3 = (x-3)(x^2+1)$.

The quadratic term does not factor into linear terms with integral coefficients, so I will not factor further.

Either $(x-3) = 0$ or $(x^2+1) = 0$. Thus $x = 3$ is one solution, and we can (try to) find others with the quadratic formula, with $a = 1, b = 0, c = 1$:

$$x = \frac{\pm\sqrt{(0)^2 - 1}}{2} = \frac{\pm\sqrt{-1}}{2}??$$

NO, no real solutions of $(x^2+1) = 0$; in fact, we could have (attempted to) solve it more easily by subtracting -1 from both sides, to get $x^2 = -1$, IMPOSSIBLE for x real.

Our only solution is $x = 3$. ∎

You can also use long division to simplify ratios of polynomials.

Example 40. Rewrite the following as a sum of a polynomial and a ratio of polynomials with the degree of the numerator less than the degree of the denominator:

$$\frac{x^4 - 2x^2 + x - 19}{x^2 + 1}.$$

Solution.

$$\begin{array}{r} x^2 - 3 \\ x^2+1 \overline{\big) x^4 - 2x^2 + x - 19} \\ -(x^4 + x^2) \\ \hline -3x^2 + x - 19 \\ -(-3x^2 - 3) \\ \hline x - 16 \end{array}$$

We conclude that

$$\frac{x^4 - 2x^2 + x - 19}{x^2 + 1} = x^2 - 3 + \frac{x - 16}{x^2 + 1}.$$

∎

41. Rational inequalities. If you wish to solve

$$R(x) \equiv \frac{p(x)}{q(x)} \le 0,$$

where $p(x)$ and $q(x)$ are polynomial expressions, first factor $p(x)$ and $q(x)$:

$$\frac{(x - a_1)(x - a_2)\cdots(x - a_n)}{(x - b_1)(x - b_2)\cdots(x - b_m)} \le 0;$$

the numbers $a_1, ..., a_n, b_1, ..., b_m$ are called *key points.*

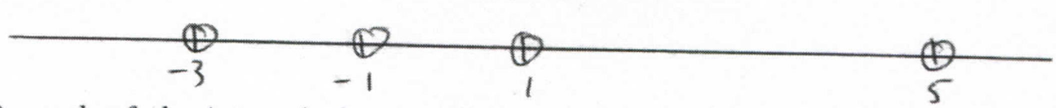
For x in the interval between two consecutive key points, $R(x)$ is either always positive or always negative. To find out which (positive or negative), plug in a point in the interval.

Examples 42. Solve each of the following.

a. $\frac{(z-1)(z+1)^2}{(z-5)(z+3)} \le 0.$

b. $\frac{x^2-6x+9}{x^2-1} > 0.$

Solutions. a. The key points here are $1, -1, 5$ and -3.



On each of the intervals $(-\infty, -3), (-3, -1), (-1, 1), (1, 5)$ and $(5, \infty)$, the expression $\frac{(z-1)(z+1)^2}{(z-5)(z+3)}$ is either always positive or always negative. For each interval, I will plug in a point to determine which it is.

$(-\infty, -3)$: I'll plug in $z = -4$:

$$\frac{(-4-1)(-4+1)^2}{(-4-5)(-4+3)} = \frac{(-5)(-3)^2}{(-9)(-1)} = \frac{-45}{9} = -5 < 0.$$

NEGATIVE on $(-\infty, -3)$.

$(-3, -1)$: I'll plug in $z = -2$:

$$\frac{(-2-1)(-2+1)^2}{(-2-5)(-2+3)} = \frac{(-3)(-1)^2}{(-7)(1)} = \frac{-3}{-7} = \frac{3}{7} > 0.$$

POSITIVE on $(-3, -1)$.

$(-1, 1)$: I'll plug in $z = 0$:

$$\frac{(0-1)(0+1)^2}{(0-5)(0+3)} = \frac{(-1)(1)^2}{(-5)(3)} = \frac{1}{5} > 0.$$

POSITIVE on $(-1, 1)$.

$(1, 5)$: I'll plug in $z = 2$:

$$\frac{(2-1)(2+1)^2}{(2-5)(2+3)} = \frac{(1)(3)^2}{(-3)(5)} = -\frac{3}{5} < 0.$$

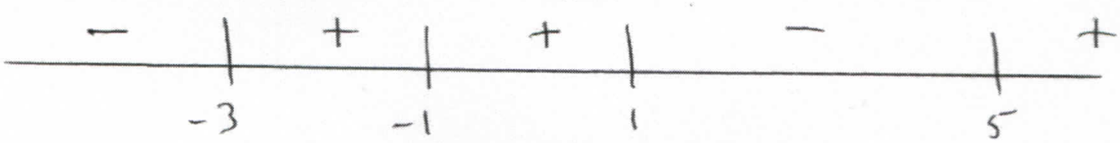
NEGATIVE on $(1, 5)$.

$(5, \infty)$: I'll plug in $z = 6$:

$$\frac{(6-1)(6+1)^2}{(6-5)(6+3)} = \frac{(5)(7)^2}{(1)(9)} > 0.$$

POSITIVE on $(5, \infty)$.

Let's put all this oscillation on a number line.



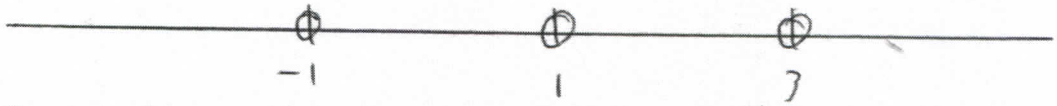
Staring at the picture we just drew tells us that the expression $\frac{(z-1)(z+1)^2}{(z-5)(z+3)}$ is negative in $(-\infty, -3)$ and $(1, 5)$. Since we're also allowing the expression to equal zero, we throw in $z = 1$ and $z = -1$ (I set the numerator—not the denominator—equal to zero).

So our solution is $\{z \mid z < -3, z = -1 \text{ or } 1 \leq z < 5\}$.

b. First factor both the numerator and denominator:

$$\frac{(x-3)^2}{(x+1)(x-1)}.$$

Thus our key points are 3 and ± 1.



On each of $(-\infty, -1), (-1, 1), (1, 3)$ and $(3, \infty)$, $\frac{(x-3)^2}{(x+1)(x-1)}$ is either always positive or always negative.

$(-\infty, -1)$: I'll plug in $x = -2$, to get

$$\frac{(-2-3)^2}{(-2+1)(-2-1)} = \frac{25}{3} > 0.$$

POSITIVE on $(-\infty, -1)$.

$(-1, 1)$: I'll plug in $x = 0$, to get

$$\frac{(0-3)^2}{(0+1)(0-1)} = -9 < 0.$$

NEGATIVE on $(-1, 1)$.

$(1, 3)$: I'll plug in $x = 2$, to get

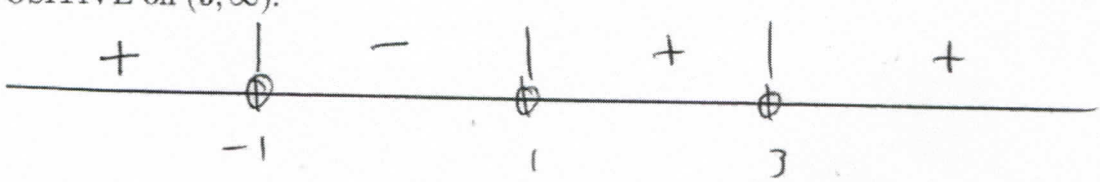
$$\frac{(2-3)^2}{(2+1)(2-1)} = \frac{1}{3} > 0.$$

POSITIVE on $(1, 3)$.

$(3, \infty)$: I'll plug in $x = 4$, to get

$$\frac{(4-3)^2}{(4+1)(4-1)} = \frac{1}{15} > 0.$$

POSITIVE on $(3, \infty)$.



Since we want our expression to be > 0, our answer is $\{x \mid x < -1,\ 1 < x < 3 \ \text{ or } \ x > 3\}$. ∎

43. Laws of exponents: (assume a is positive)

(1) $a^x a^y = a^{x+y}$

(2) $(a^x)^y = a^{xy}$

Note: You should know how to show (from 43(1) and (2)) that $a^0 = 1, a^{-x} = \frac{1}{a^x}$ and $a^{\frac{1}{n}}$ is the n^{th} root of a.

Examples 44. Simplify each of the following. a. $\sqrt{3^4}$ b. $\frac{1}{2^{-3}}$ c. $\left((\sqrt{2})^{\sqrt{2}}\right)^{\sqrt{2}}$ d. $\frac{4^{-4}}{8^{-2}}$ e. $(3^{-1+\sqrt{5}})^2(9^{1-\sqrt{5}})$.

Solutions. a. This is $(3^4)^{\frac{1}{2}} = 3^{4\cdot\frac{1}{2}} = 3^2 = 9$, by 43(2).

b. This is $(2^{-3})^{-1} = 2^{(-3)(-1)} = 2^3 = 8$, by 43(2).

c. This is $(\sqrt{2})^{(\sqrt{2})(\sqrt{2})} = (\sqrt{2})^2 = 2$.

d. This is $4^{-4}\left(8^{-2}\right)^{-1} = 4^{-4}8^2$, by 43(2), which equals $(2^2)^{-4}(2^3)^2 = (2^{-8})(2^6)$, again by 43(2), which equals $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$, by 43(1).

e. This is

$$\left(3^{-1+\sqrt{5}}\right)^2\left((3^2)^{1-\sqrt{5}}\right) = \left(3^{-2+2\sqrt{5}}\right)\left(3^{2-2\sqrt{5}}\right),$$

by 43(2), which equals $3^{(-2+2\sqrt{5}+2-2\sqrt{5})} = 3^0 = 1$, by 43(1). ∎

Examples 45. Solve each of the following.

a. $2^x = \frac{1}{4}$.

b. $38 = 2 + 4(3^{\frac{t}{4}})$.

c. $3^{2x+5} = 9\sqrt{3}$.

d. $\frac{2^{x^2}}{(2^x)^2} = 8$.

Solutions. Use the fact that, if $b^a = b^c$, then a must equal c.

a. $2^x = \frac{1}{4} = 4^{-1} = (2^2)^{-1} = 2^{-2}$, by 43(2), so $x = -2$.

b. I want to isolate the $3^{\frac{t}{4}}$. So I subtract 2 from both sides: $36 = 4(3^{\frac{t}{4}})$, then divide both sides by 4:

$$3^{\frac{t}{4}} = 9 = 3^2,$$

so $\frac{t}{4} = 2$, thus $t = 8$.

c. $3^{2x+5} = 9\sqrt{3} = 3^2 3^{\frac{1}{2}} = 3^{\frac{5}{2}}$, by 43(1). Equating exponents gives us $2x + 5 = \frac{5}{2}$, so that $x = \frac{1}{2}(\frac{5}{2} - 5) = -\frac{5}{4}$.

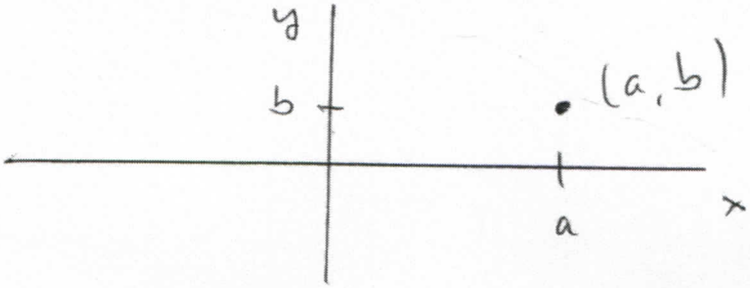
d. Here we can write each side as a power of 2, then equate the exponents.

$$2^3 = 8 = \frac{2^{x^2}}{(2^x)^2} = 2^{x^2}\left((2^x)^2\right)^{-1} = 2^{x^2}2^{-2x} = 2^{x^2-2x},$$

so $x^2 - 2x = 3$ and we have a quadratic equation to solve. Write the equation as $x^2 - 2x - 3 = 0$ and factor: $(x-3)(x+1) = 0$, so $x = 3$ or -1. ∎

46. We represent ordered pairs (a, b) of real numbers as points in a plane as follows. A horizontal line, called the **x axis**, and a vertical line, called the **y axis**, are drawn on a plane, such as a piece of paper. Place (a, b) at the point a units to the right of the y axis and b units above the x axis.

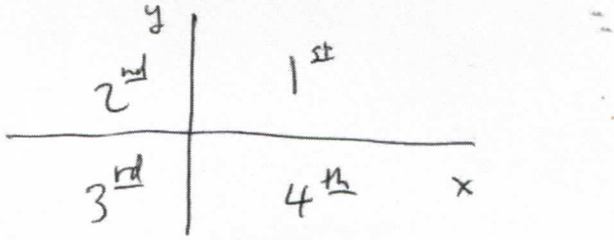
Note that moving a negative number of units to the right means you're moving to the left; e.g., -5 units to the right means 5 units to the left. Similarly, -5 units above the x axis means 5 units below the x axis.



The **coordinates** of the point P are (a, b); its x-coordinate, or abscissa, is a and its y-coordinate, or ordinate, is b.

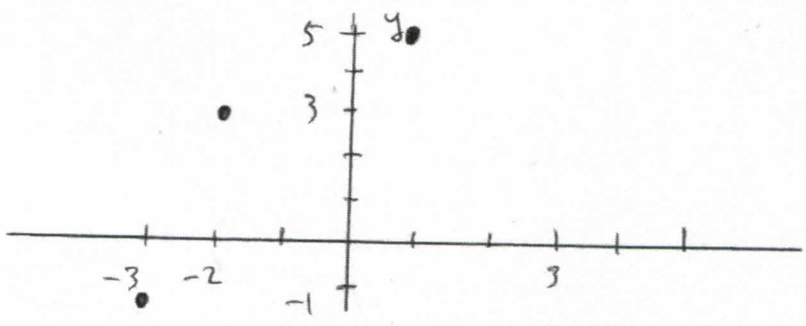
The set of all these ordered pairs $\{(a, b) \mid a, b \text{ are real } \}$ is called the **Cartesian plane**. This method of describing points in the plane is called **Cartesian coordinates** (coordinate system) or **rectangular coordinates** (coordinate system).

The **origin** is $(0, 0)$, at the intersection of the x and y axes. The **first quadrant** is $\{(a, b) \mid a > 0, b > 0\}$; the **second quadrant** is $\{(a, b) \mid a < 0, b > 0\}$; The **third quadrant** is $\{(a, b) \mid a < 0, b < 0\}$; The **fourth quadrant** is $\{(a, b) \mid a > 0, b < 0\}$.



Examples 47. Plot each of the following ordered pairs in the Cartesian plane and identify which quadrant each is in: $(1, 5), (-2, 3), (-3, -1)$.

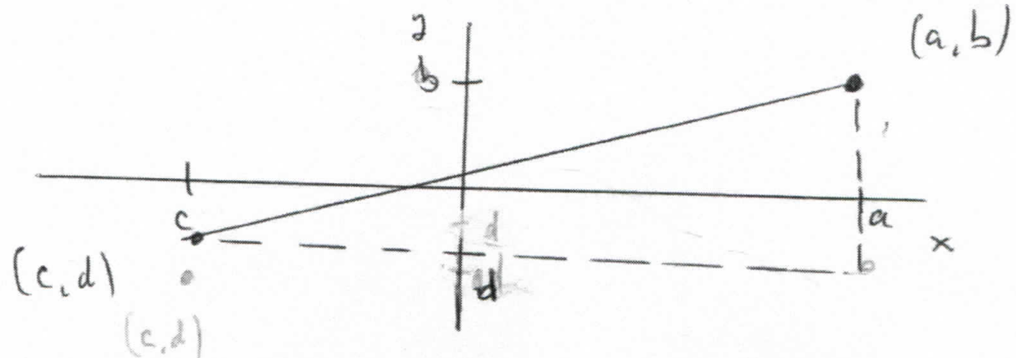
Solutions.



The point $(1, 5)$ is in the first quadrant; $(-2, 3)$ is in the second quadrant; $(-3, -1)$ is in the third quadrant. ∎

48. Distance in the Cartesian plane. The distance between two points (a, b) and (c, d) is

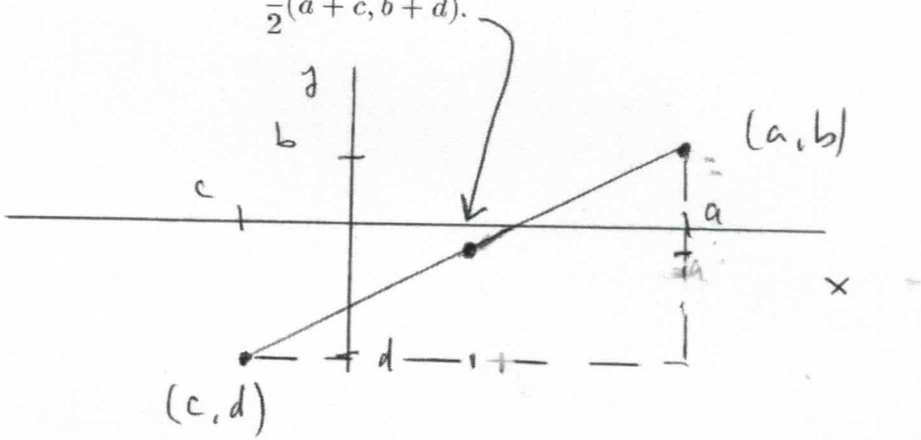
$$\sqrt{(a-c)^2 + (b-d)^2}.$$



This follows from the Pythagorean theorem, since we have a right triangle with one leg of length $|a - c|$ and the other leg of length $|b - d|$, while the hypotenuse is exactly the distance we crave.

49. The **midpoint** of the line between (a, b) and (c, d) is

$$\frac{1}{2}(a + c, b + d).$$



Compare this with 9.

Examples 50. a. What is the distance between $(-1, 3)$ and $(2, 1)$?
b. Find the midpoint of the line between $(-3, 1)$ and $(2, -5)$.

Solutions. a.

$$\sqrt{(-1-2)^2 + (3-1)^2} = \sqrt{9+4} = \sqrt{13}.$$

b.

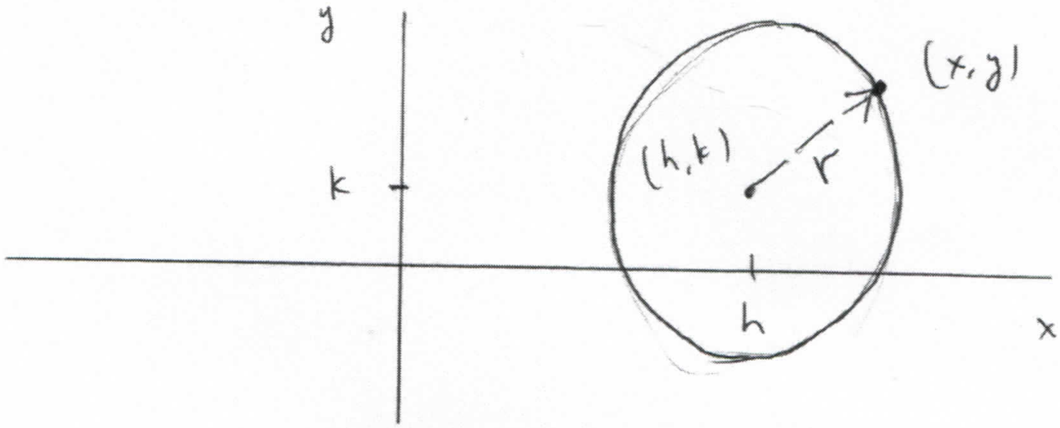
$$\frac{1}{2}(-3 + 2, 1 + (-5)) = (-\frac{1}{2}, -2).$$

∎

51. Graph of circle. A circle of radius r, centered at (h,k), is the set of all points (x,y) that satisfy the equation

$$(x-h)^2+(y-k)^2=r^2.$$

This follows from the distance formula 48, applied to (x,y) and (h,k); we want the set of all points (x,y) such that the distance from (x,y) to (h,k) is r.



Examples 52. a. Find the equation of the circle centered at $(1,-2)$, with radius 5.

b. Find the center and radius of the circle satisfying the equation

$$2x^2-8x+2y^2+6y+12=0.$$

Solutions. a. Apply 51, with $h=1, k=-2$ and $r=5$:

$$(x-1)^2+(y+2)^2=25.$$

b. Let's get the 12 out of the way:

$$(2x^2-8x)+(2y^2+6y)=-12.$$

To make our equation look like 51, complete the square (see 29) in both $(2x^2-8x)$ and $(2y^2+6y)$; let's start by dividing both sides of the equation by 2:

$$(x^2-4x)+(y^2+3y)=-6,$$

so

$$\left(x^2-4x+\left(\frac{4}{2}\right)^2\right)+\left(y^2+3y+\left(\frac{3}{2}\right)^2\right)=-6+\left(\frac{4}{2}\right)^2+\left(\frac{3}{2}\right)^2=\frac{1}{4},$$

or

$$(x-2)^2 + (y+\frac{3}{2})^2 = \frac{1}{4}.$$

Staring at the equation in 51, this tells us that $h = 2, k = -\frac{3}{2}$ and $r = \sqrt{\frac{1}{4}} = \frac{1}{2}$; that is, the center of this circle is $(2, -\frac{3}{2})$ and the radius is $\frac{1}{2}$. ▮

53. Distance, Speed and Time. If you travel at a constant speed v, for a time t, then your distance d is given by

$$d = vt.$$

Examples 54. a. Suppose you drive at 60 miles per hour for 5 hours and 20 minutes. How far will you travel?

b. If you leave your house at 5 AM, and travel at 40 miles per hour towards the Syrupy Soda Shoppe 150 miles from your house, at what time will you arrive?

c. Suppose Li'l Abner has a 4-kilometer head start on Daisy Mae at the Sadie Hawkins Day race, and is running away at a constant speed of 10 kilometers per hour. At what speed should Daisy Mae run if she wants to overtake Li'l Abner in 3 hours?

Solutions. a. 5 hours and 20 minutes is $\frac{16}{3}$ hours, so our distance is $(60)(\frac{16}{3})$ miles, equal to 320 miles.

b. Let t be the time you spend travelling. Then

$$150 \text{ miles} = \left(40\frac{\text{miles}}{\text{hour}}\right)t,$$

so $t = \frac{150}{40}$ hours $= \frac{15}{4}$ hours, which is 3 hours and 45 minutes. This is added to 5 AM, to get the time of 8:45 AM.

c. Let s be Daisy Mae's speed. She is overtaking Li'l Abner at a speed of $(s-10)$ kilometers per hour, and has 4 kilometers to gain in 3 hours, so I apply 51 with $d = 4, v = (s-10)$ and $t = 3$, to get

$$4 = (s-10)3 \rightarrow \frac{4}{3} = (s-10) \rightarrow s = 10 + \frac{4}{3} = \frac{34}{3}.$$

The answer is $\frac{34}{3}$ kilometers per hour. ▮

REVIEW HOMEWORK

1. Find the distance between 2 and -7.

2. Solve $\frac{6}{7} = \frac{3}{x}$.

3. Solve $\frac{1}{2x} - 1 = \frac{1}{4}$.

4. Solve $3 - \frac{y}{2} = 2y - 4$.

5. Suppose a is a number whose absolute value is 7. What is the absolute value of $-a$?

6. Solve and graph:
a. $2x - 7 \geq -3x + 9$.
b. $|2x + 4| > 5$.
c. $3 + |x - 1| \leq 2$.
d. $-|1 - \frac{x}{4}| + 10 \geq 4$.

7. Find all x such that the distance from x to -5 is less than 3. Graph your solution.

8. For each of the following, either factor or assert that it is irreducible over the integers.
a. $(27z^3 - 8x^6)$.
b. $4x^2 - 12x + 9$.
c. $y^2 - 3y - 4$.
d. $2x^2 - x - 3$.
e. $x^2 + 4x - 3$.

9. Simplify: a. $(\sqrt{2})^6$ b. $\frac{(3^{2-\sqrt{2}})^4}{9}$ c. $(3^{2+\sqrt{5}})^2(9^{1-\sqrt{5}})$ d. $\left((\sqrt{2})^{-2}\right)^{-1}$
e. $\frac{4+\sqrt{8}}{2}$ f. $\frac{a}{\sqrt{a}}$ g. $\frac{(a^5)^2}{a^6\sqrt{a}}$ h. $\frac{1}{\sqrt{a}}(a^2 - 3\sqrt{a})$.

10. Complete the square.
a. $x^2 + 9x - 1$.
b. $2y^2 - 20y + 9$.

11. Solve by completing the square.
a. $x^2 = 8x + 10$.
b. $3y^2 - 18y + 1 = 0$.

12. Solve by factoring: $x^2 - 7x = 18$.

13. Solve $2y^2 + 7y = 1$.

14. Without solving, determine how many real solutions the equation has.

a. $9z^2 + 12z = -4$.

b. $3x^2 + x + 1 = 0$.

15. Factor $x^3 - 7x + 6$.

16. Write $\frac{x^3+x^2+1}{x+1}$ as the sum of a polynomial and a fraction with the degree of the numerator less than the degree of the denominator.

17. Solve and graph.

a. $(z+1)^2(z-1)(z-5)(z^2+1) > 0$.

b. $x^2 + x \leq 6$.

c. $\frac{y(1-y^2)}{(y^2-4)} \leq 0$.

18. Solve

a. $4(3^x) = 36$.

b. $2(4^x) + 1 = 2$.

19. Find the distance between $(-5, 1)$ and $(2, 0)$ and the midpoint of the line between them.

20. If $(1, 2)$ is the midpoint of the line between $(-4, -1)$ and (a, b), what are a and b?

21. Suppose I start at noon 50 miles behind a bus going 60 miles per hour. Three hours later, I'm 10 miles behind the bus. How fast am I going? When will I catch the bus?

22. Find the equation of the circle centered at $(1, 4)$ with radius 2.

23. Find the center and radius of the circle satisfying the equation

$$2x^2 - 8x + 2y^2 + 6y - 1 = 0.$$

Graph the circle.

ANSWERS

1. 9. **2.** $\frac{7}{2}$. **3.** $\frac{2}{5}$. **4.** $\frac{14}{5}$. **5.** 7.

6. a. $[\frac{16}{5}, \infty)$; also known as $x \geq \frac{16}{5}$.

b. $x > \frac{1}{2}$ or $x < -\frac{9}{2}$.

c. no solutions.

d. $[-20, 28]$; also known as $-20 \leq x \leq 28$.

7. $(-8, -2)$; also known as $-8 < x < -2$.

8. a. $(3z - 2x^2)(9z^2 + 6zx^2 + 4x^4)$ (this is a difference of two cubes $(a^3 - b^3)$, with $a = 3z, b = 2x^2$).
b. $(2x - 3)^2$.
c. $(y - 4)(y + 1)$.
d. $(2x - 3)(x + 1)$.
e. Irreducible over the integers.

9. a. 8 b. $3^{6-4\sqrt{2}}$ c. 3^6 d. 2 e. $(2 + \sqrt{2})$
f. $\sqrt{a}$ g. $a^3\sqrt{a}$ or $a^{\frac{7}{2}}$ h. $(a\sqrt{a} - 3)$.

10. a. $(x + \frac{9}{2})^2 - \frac{85}{4}$.
b. $2(y - 5)^2 - 41$.

11. a. $(x - 4)^2 = 26 \rightarrow x = 4 \pm \sqrt{26}$.
b. $(y - 3)^2 = \frac{26}{3} \rightarrow y = 3 \pm \sqrt{\frac{26}{3}}$.

12. $(x - 9)(x + 2) = 0 \rightarrow x = 9$ or -2.

13. $y = \frac{-7 \pm \sqrt{57}}{4}$.

14. a. one real solution b. no real solutions.

15. $(x-1)(x+3)(x-2)$.

16. $x^2 + \frac{1}{x+1}$.

17. a. $\{z \mid z < -1, -1 < z < 1 \text{ or } z > 5\}$.

b. $[-3, 2]$.

c. $\{y \mid -2 < y \leq -1 \text{ or } 0 \leq y < 1 \text{ or } y > 2\}$.

18. a. $x = 2$ b. $x = -\frac{1}{2}$.

19. distance is $5\sqrt{2}$; midpoint is $(-\frac{3}{2}, \frac{1}{2})$.

20. $a = 6, b = 5$.

21. My speed is $\frac{220}{3}$ miles per hour; I'll catch the bus at $3:45$ p.m.

22. $(x-1)^2 + (y-4)^2 = 4$.

23. center is $(2, -\frac{3}{2})$, radius is $\frac{3\sqrt{3}}{2}$.