Concise Lecture Notes Series

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Fun with

Functions

Dr. Ralph deLaubenfels

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CONCISE PRECALCULUS

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CONCISE INTRODUCTION

The thrust of this book and class is functions. As an example, suppose you are trying to decide how much seed to buy for a disc-shaped field. You have one quantity, the radius of the disc, that you can measure. You have another quantity, the area of the disc, that you need to know. You can get what you want from what you know with the formula

$$A = \pi r^2 \quad (r = \text{ radius}, A = \text{ area }).$$

We say that A is a *function* of r; we make this relationship explicit by writing $A(r)$, which reads "A *of* r." In general, a function relates what you can measure (this is called the *domain*) to what you want to know (this is called the *range*).

ALGEBRA YOU SHOULD KNOW

1.1. set terminology;

1.2. equation terminolgy;

1.3. distributive law;

1.4. natural numbers, integers, rational numbers, irrational numbers, and **real numbers**;

1.5. intervals;

1.6. midpoints;

1.7 absolute value;

1.8. distance between two real numbers;

1.9. how to solve **linear equations and inequalities**;

1.10. how to solve inequalities with absolute values in them;

1.11. factoring and completing the square, for **quadratic expressions**;

1.12. how to recognize and factor **perfect squares, perfect cubes** and **differences of two squares**;

1.13. how to use factorization to solve equations;

1.14. how to use completing the square to solve quadratic equations;

1.15. quadratic formula;

1.16. long division of polynomials (and using to simpify rational functions);

1.17. how to solve **rational inequalities**;

1.18 laws of exponents;

1.19. the **Cartesian plane**;

1.20. distance in the Cartesian plane;

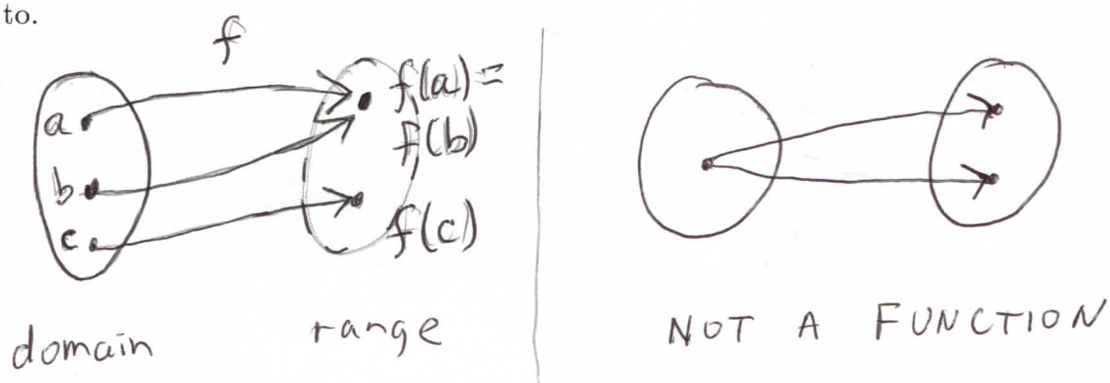
1.21. graph of a circle;

1.22. distance, speed and time.

II. FUNCTIONS

2.1. A function is a rule that assigns to each point in the *domain* a unique point in the *range*, which is $\{f(x) \,|\, x \text{ is in the domain of } f\}$.

Write $f(x)$, in words, "f of x," for the *image* of x, the point that x is assigned to.



The function I just drew could also be represented by a table.

x	a	b	c
$f(x)$	1	1	2

Examples 2.2. a. Start 30 miles from Athens, and travel away from Athens at 20 miles per hour; let $d(t) \equiv$ your distance from Athens t hours after starting.

b. A plumber charges a flat fee of 30 dollars, plus 20 dollars per hour or fraction thereof; let $P(t)$ be what she/he charges for t hours of work.

Both these functions can be written down explicitly.

a. $d(t) = 30 + 60t$, by 1.51.

b. At first glance, this looks very similar to a; perhaps $P(t)$ equals $30 + 20t$? NO, this is probably not what is meant; for one minute of work, for example, you would then be charged 30 and $\frac{1}{3}$ dollars, or 30 dollars and $33\frac{1}{3}$ cents; difficulties arise when you try to find coins whose value equals $\frac{1}{3}$ of a cent.

What is actually meant by the charges in b is $P(t) = 30$, for $0 \leq t < 1$, $P(t) = 50$, for $1 \leq t < 2$, ... We write it the following way.

$$P(t) = \begin{array}{ll} 30 & 0 \leq t < 1 \\ 50 & 1 \leq t < 2 \\ 70 & 2 \leq t < 3 \\ 90 & 3 \leq t < 4 \\ . & \ldots\ldots \\ . & \ldots\ldots \\ . & \ldots\ldots \end{array}$$

▮

The word "function" is regular English; in Example 2.2(a), you would say "distance is a function of time." In other words, if you know the time, you know the distance. The idea is that time is something you can immediately measure (by looking at your watch), and distance from Athens is something you want to know, for example, if a thermonuclear device is being set off in Athens.

2.3. Plugging in. It must be emphasized that when one writes down a function such as $f(x) = x^2$, the x is arbitrary. The function f should be thought of as the act of squaring, or a set of instructions: Take whatever is inside the parentheses, and square it.

Example 2.4. Suppose $f(x) = x^2$ and $g(x) = 3x + 5$. Find each of the following: a. $f(3)$ b. $f(\sqrt{2})$ c. $f(y)$ d. $f(g(-1))$ e. $f(g(x))$ f. $g(f(x))$ g. $f(\sqrt{x})$ h. $f(-\sqrt{x})$ i. $f(\frac{1}{x+1})$ j. $f(f(z))$.

Solutions. a. $3^2 = 9$ b. $(\sqrt{2})^2 = 2$ c. y^2 d. $f(3(-1)+5) = f(2) = 2^2 = 4$ e. $f(3x+5) = (3x+5)^2$ f. $g(x^2) = 3x^2 + 5$ g. $(\sqrt{x})^2 = x$ h. $(-\sqrt{x})^2 = x$ i. $(\frac{1}{x+1})^2 = \frac{1}{(x+1)^2}$ j. $f(z^2) = (z^2)^2 = z^4$. ∎

Example 2.5. Suppose

$$f(y) = \begin{matrix} y^2 & -2 < y < 0 \\ 3 & 0 < y \le 1 \\ \sqrt{y} & 1 < y \le 5 \end{matrix}$$

and

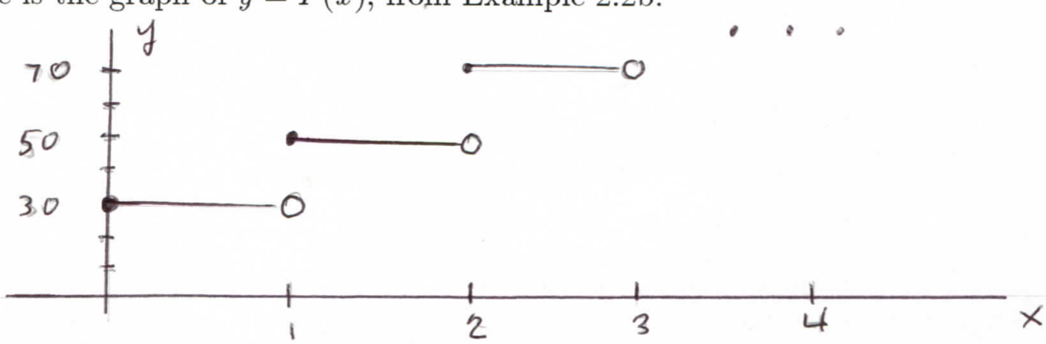
$$g(z) = \begin{matrix} -z^3 & z \le -1 \\ \frac{1}{1+z} & -1 < z \le 1 \\ 2^z & 3 \le z < 7 \end{matrix}$$

Find (if possible) each of the following: a. $f(3)$ b. $f(g(-4))$ c. $g(f(1))$ d. $g(-2)$ e. $f(f(4))$ f. $g(2)$ g. $f(-f(4))$.

Solutions. a. $f(3) = \sqrt{3}$. b. $f(g(-4)) = f(64)$, which is not defined; your answer should be "undefined." c. $g(f(1)) = g(3) = 2^3 = 8$. d. $g(-2) = 8$. e. $f(f(4)) = f(\sqrt{4}) = f(2) = \sqrt{2}$. f. undefined. g. $f(-f(4)) = f(-\sqrt{2}) = (-\sqrt{2})^2 = 2$. ∎

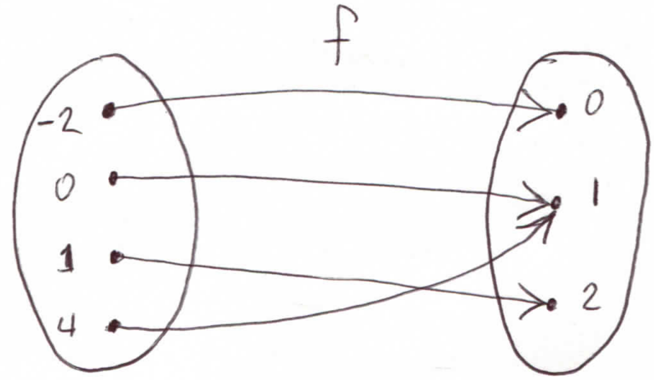
2.6. The **graph** of a function f is $\{(x, y) \mid x \text{ is in the domain of } f,\ y = f(x)\}$.

Here is the graph of $y = P(x)$, from Example 2.2b.



Here are four representations of the same function, f; the last two are graphs.

(1)

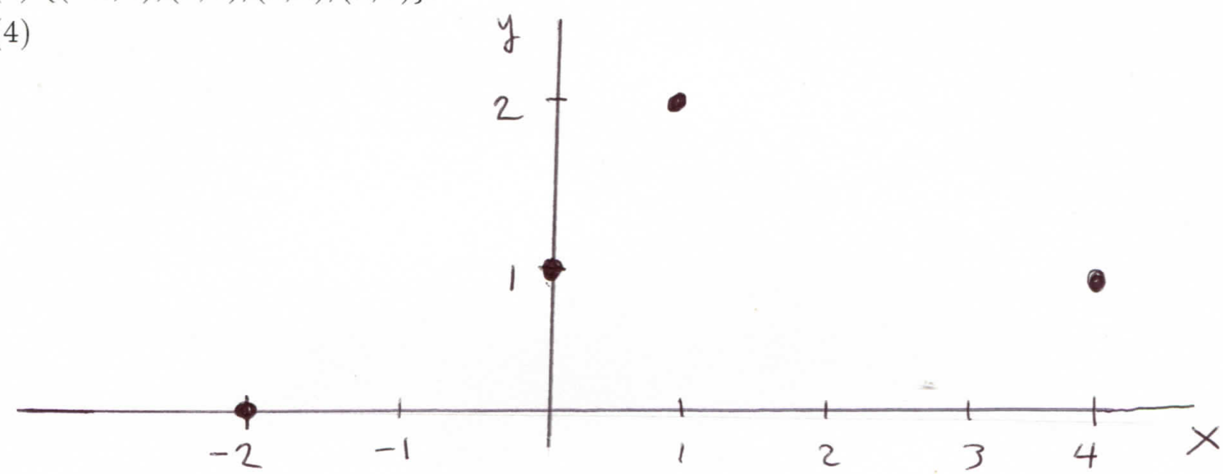


(2)

x	-2	0	1	4
$f(x)$	0	1	2	1

(3) $\{(-2,0),(0,1),(1,2),(4,1)\}$.

(4)

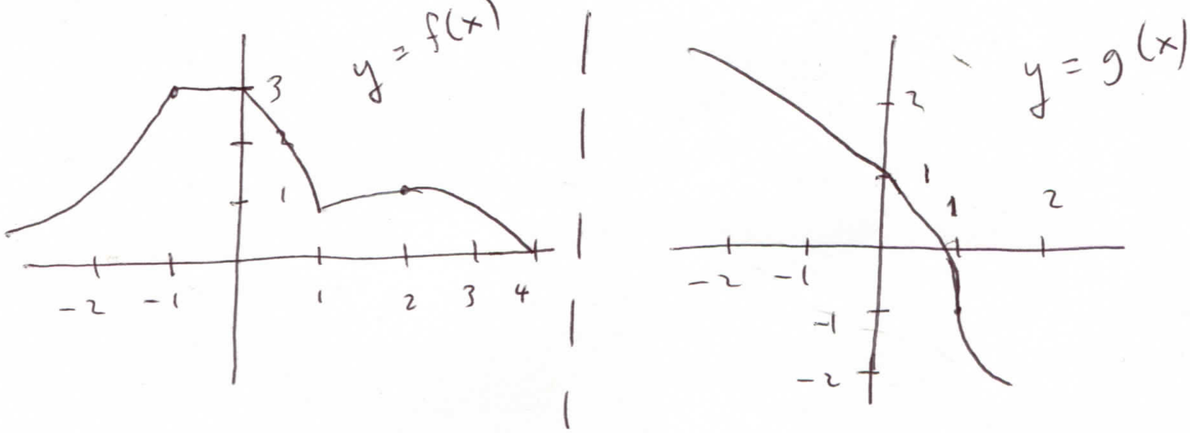


2.7. The **vertical line test** says that a graph is the graph of a function if and only if any vertical line crosses the graph at most once.

2.8. An **x intercept** of the graph of a function f is a value of x such that $f(x) = 0$. In other words, the graph of f crosses the x axis at an x-intercept. The **y intercept** is $f(0)$. The graph of f crosses the y axis at its y intercept.

It is possible for the graph of a function to have no intercepts.

Example 2.9. Use the following graphs to estimate a. $f(2)$ b. $g(1)$ c. $f(g(1))$ d. all intercepts of the graph of f.



Solutions. a. 1 b. -1 c. 3 d. y intercept of 3, x intercept of 4. ▮

2.10. Domain angst. Even if the restrictions are not explicitly written down in the definition of the domain of a function, you must worry about and avoid

(1) A zero in a denominator, and

(2) A negative number inside a square root.

Examples 2.11. Find the domains of each of the following functions.

a. $f(x) = \sqrt{x+2}$

b. $g(x) = \frac{1}{x^2+x-2}$

c. $h(x) = \sqrt{x^2 - x + 2}$.

d.

$$f(x) = \begin{array}{cc} \sqrt{1+x^2} & x \leq 1 \\ \frac{1}{x} & 1 < x < 2 \\ & \\ \sqrt{16-x^2} & 2 < x \leq 5 \\ \frac{1}{\sqrt{8-x}} & x > 5 \end{array}$$

Solutions. a. By 2.10(2), we need $x + 2 \geq 0$, or $x \geq -2$, so that the domain of f is $[-2, \infty)$.

b. By 2.10(1), we must avoid $x^2 + x - 2 = 0$. The quadratic expression factors into $(x+2)(x-1)$, so we are avoiding $x = 1$ and $x = -2$. The domain of g is $\{x \mid x \neq 1 \text{ and } x \neq -2\}$.

c. Now we need $x^2 - x + 2 \geq 0$, by 2.10(2). Solve this as in 1.41: $(x-2)(x+1) \geq 0$ has key points $x = -1$ and $x = 2$, so we check the sign of $(x-2)(x+1)$ on $(-\infty, -1), (-1, 2)$ and $(2, \infty)$:

$(-\infty, -1)$: plug in $x = -2$:

$$(-2-2)(-2+1) = 4 > 0.$$

POSITIVE on $(-\infty, -1)$.

$(-1, 2)$: plug in $x = 0$:

$$(0-2)(0+1) = -2 < 0.$$

NEGATIVE on $(-1, 2)$.

$(2, \infty)$: plug in $x = 3$:

$$(3-2)(3+1) = \frac{1}{4} > 0.$$

POSITIVE on $(2, \infty)$.

The domain of h is the solution of our inequality: $\{x \mid x \leq -1 \text{ or } x \geq 2\}$.

d. Calculating domain is a negative activity. You look for trouble, then remove, from the domain, any of those troublesome points.

Since this function is defined in pieces, with each piece giving potential trouble, we need to be organized. I'm going to put an asterisk (*) by any x that must be *removed* from the domain.

First I notice that 2 is left out of our domain in the definition of f. So

$$x = 2 \qquad (*)$$

must be thrown out.

We also have to worry about 2.10(1) and (2), where the function is defined.

Since x^2 is always nonnegative, so is $1 + x^2$, thus there is no difficulty for $x \leq 1$.

For $1 < x < 2$, we must worry about x being equal to zero; but since $f(x) = \frac{1}{x}$ only when $1 < x < 2$, this is no problem.

Next, look at $\sqrt{16 - x^2}$. Solving

$$16 - x^2 \geq 0$$

as in 1.41 gives

$$-4 \leq x \leq 4.$$

Since $f(x) = \sqrt{16 - x^2}$ for $2 < x \leq 5$ (allegedly), we must throw

$$4 < x \leq 5 \qquad (*)$$

out of the domain of f, by 2.10(2).

Finally, $\frac{1}{\sqrt{8-x}}$ is defined only when $8 - x > 0$ or, equivalently, $x < 8$. Since $f(x) = \frac{1}{\sqrt{8-x}}$ (allegedly) when $x > 5$, we must also throw

$$x \geq 8 \qquad (*)$$

out of the domain, by 2.10(1) and (2).

Removing all the bad stuff we found, as marked by asterisks (*) above, gives us $\{x \mid x < 2,\ 2 < x \leq 4 \text{ or } 5 < x < 8\}$ for the domain of f. ∎

2.12. How to calculate the range of a function f: Set $y = f(x)$, and *try* to solve for x.

Examples 2.13. In each of the following, find the range of f.

a. $f(x) = 2x^2 - 6x + 1$.

b. $f(x) = \frac{3x+1}{x-2}$.

c. $f(x) = -2x + 5$, domain of f equal to $[0, \infty)$.

Solutions. a. Set $y = 2x^2 - 6x + 1$, and (try to) solve for x with the quadratic formula (1.33):

$$2x^2 - 6x + (1 - y) = 0,$$

so

$$x = \frac{1}{4}\left[6 \pm \sqrt{36 - 8(1 - y)}\right].$$

By 2.10(2), the expression we got for x is defined only when

$$0 \leq 36 - 8(1 - y) = 28 + 8y;$$

solving that inequality for y gives us

$$y \geq -\frac{28}{8} = -\frac{7}{2}.$$

That's the range of f: $[-\frac{7}{2}, \infty)$.

b. Setting $y = \frac{3x+1}{x-2}$ implies that $3x + 1 = y(x - 2) = yx - 2y$, so that $2y + 1 = yx - 3x = x(y - 3)$, and

$$x = \frac{2y + 1}{y - 3}.$$

This is defined for any y except $y = 3$ (by 2.10(1)); the range of f is $\{y \mid y \neq 3\}$.

c. If $y = -2x + 5$, then $x = \frac{1}{2}(5 - y)$. That puts no restrictions on y; however, the stated domain of f is saying that $x \geq 0$, thus

$$\frac{1}{2}(5 - y) \geq 0.$$

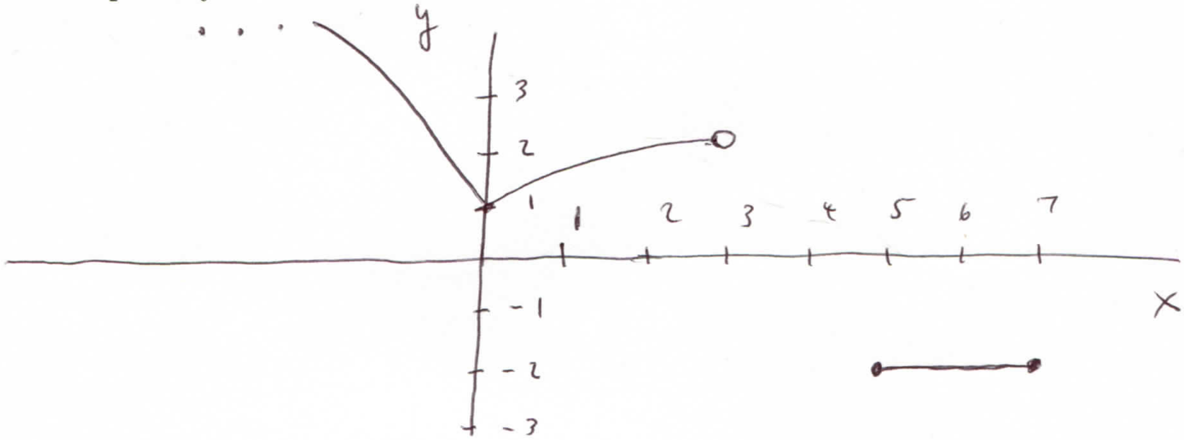
Solving that inequality for y gives us

$$y \leq 5.$$

That is, the range of f is $(-\infty, 5]$. ∎

One may identify the domain and range of a function from its graph. The domain is the shadow of the graph on the x axis, from a light above and below the graph, while the range is the shadow of the graph on the y axis, from a light on the left and right hand sides of the graph.

Example 2.14. Below is the graph of a function f. Estimate the domain, range and intercepts of f.



Solution. Domain is $\{x \mid x < 3 \text{ or } 5 \leq x \leq 7\}$, Range is $\{y \mid y \geq 1 \text{ or } y = -2\}$, the y intercept is 2 and there is no x intercept. ∎

Here are our favorite functions.

2.15. Polynomials:

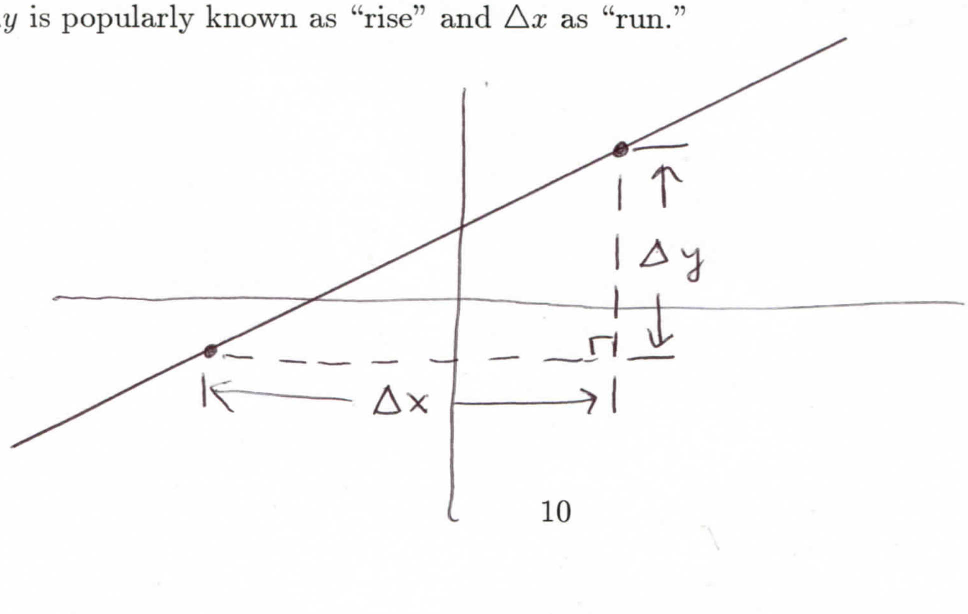
$$p(x) \equiv a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N,$$

where $a_0, a_1, ..., a_N$ are numbers; if $a_N \neq 0$, then N is the *degree* of the polynomial.

A 0^{th}-degree polynomial is a **constant function**. A 1^{st}-degree polynomial is a **linear function**. The graph of a first-degree polynomial is a straight line. The **slope** of a straight line is

$$\frac{\Delta y}{\Delta x} \equiv \frac{y_1 - y_2}{x_1 - x_2}, \qquad \text{read "delta"} \tag{2.16}$$

where (x_1, y_1) and (x_2, y_2) are any two points on the line (Δ means "change in"). Δy is popularly known as "rise" and Δx as "run."



A straight line is determined by any two points on the line. When a linear function is written $f(x) = mx + b$, where m and b are numbers, the slope of its graph is m and the y intercept is b. When two lines, with slopes m_1 and m_2, are parallel, then $m_1 = m_2$; if they are perpendicular, then $m_1 m_2 = -1$.

Example 2.17. In each of the following, find the linear function whose graph:

a. goes through $(1, 5)$ and has slope 7;

b. goes through $(2, 3)$ and $(-1, 4)$;

c. goes through $(-3, 0)$ and is parallel to $2y + 3x = 19$;

d. goes through $(-6, -12)$ and is perpendicular to $3x - y = 0$;

e. goes through $(1, 2)$ and is perpendicular to $x = 5$;

f. has x-intercept 2 and goes through $(-1, -2)$.

Solutions.

a. Let (x, y) be an arbitrary point on the line. Apply (2.16) to the points (x, y) and $(1, 5)$:

$$7 = \text{ slope } = \frac{y - 5}{x - 1}.$$

Solving for y gives us $y = 5 + 7(x - 1) = 7x - 2$ as the graph of the desired linear function $f(x) = 7x - 2$.

b. First apply (2.16) to the points $(2, 3)$ and $(-1, 4)$ to get the slope:

$$\text{slope } = \frac{4 - 3}{(-1) - 2} = -\frac{1}{3}.$$

Now, as in a, apply (2.16) to $(2, 3)$ and an arbitrary point (x, y) on the line:

$$-\frac{1}{3} = \frac{y - 3}{x - 2}$$

so that $y = 3 - \frac{1}{3}(x - 2) = -\frac{1}{3}x + \frac{11}{3}$ is the graph of the desired linear function $f(x) = -\frac{1}{3}x + \frac{11}{3}$.

c. The line it's parallel to is $y = -\frac{3}{2}x + \frac{19}{2}$, which has slope $-\frac{3}{2}$, thus our line also has slope $-\frac{3}{2}$, so our line satisfies

$$-\frac{3}{2} = \frac{y - 0}{x + 3},$$

thus $y = -\frac{3}{2}(x + 3) = -\frac{3}{2}x - \frac{9}{2}$, and our linear function is $f(x) = -\frac{3}{2}x - \frac{9}{2}$.

d. The line it's perpendicular to can be written $y = 3x$, thus has slope 3, so our line has slope $-\frac{1}{3}$, thus it satisfies

$$-\frac{1}{3} = \frac{y + 12}{x + 6},$$

or $y = -12 - \frac{1}{3}(x + 6) = -\frac{1}{3}x - 14$, and our linear function is $f(x) = -\frac{1}{3}x - 14$.

e. $x = 5$ is a vertical line; thus our line is horizontal. Horizontal lines have the form $y = b$, for some number b; since $(1, 2)$ is on our line, $b = 2$; that is, our line is $y = 2$, and our function is the constant function $f(x) = 2$.
f. An x-intercept of 2 means that $(2, 0)$ is on the line. Thus we proceed as in b, getting the line through $(2, 0)$ and $(-1, -2)$:

$$\text{slope} = \frac{-2-0}{-1-2} = \frac{2}{3}.$$

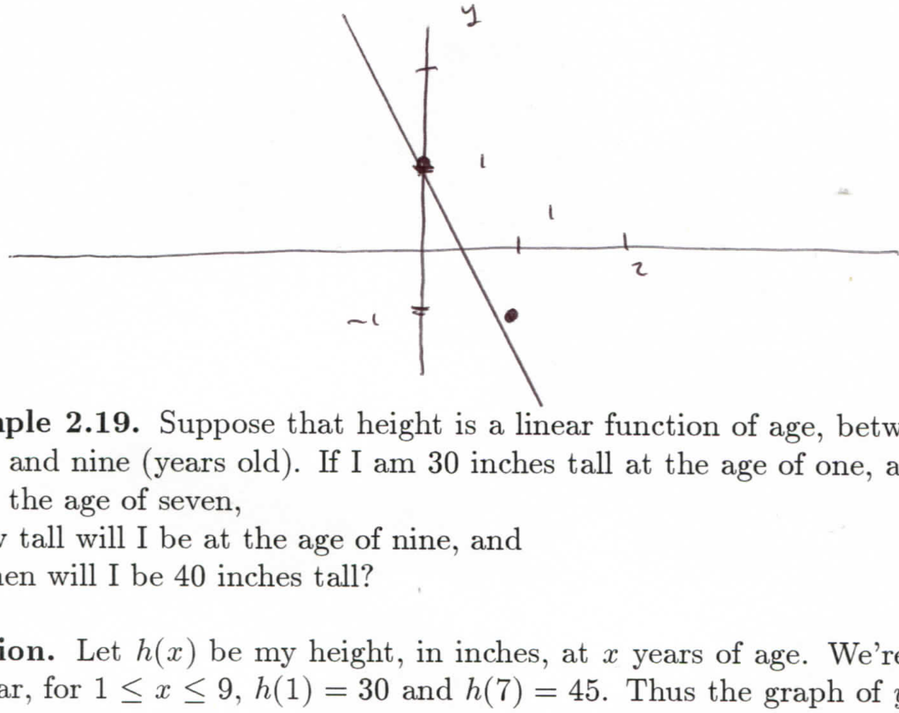
If (x, y) is on the line, then

$$\frac{2}{3} = \frac{y-0}{x-2},$$

thus $y = \frac{2}{3}(x-2)$ is the graph of our desired linear function $f(x) = \frac{2}{3}(x-2)$. ∎

Example 2.18. Graph $f(x) = 1 - 2x$.

Solution. Since f is a linear function, its graph is a line. Thus all I need is two points on the line $y = 1 - 2x$. When $x = 0, y = f(0) = 1$; when $x = 1, y = f(1) = -1$. So $(0, 1)$ and $(1, -1)$ are two points on our line; graph those two points, then draw the line between them.



Example 2.19. Suppose that height is a linear function of age, between the ages of one and nine (years old). If I am 30 inches tall at the age of one, and 45 inches tall at the age of seven,
a. how tall will I be at the age of nine, and
b. When will I be 40 inches tall?

Solution. Let $h(x)$ be my height, in inches, at x years of age. We're told that h is linear, for $1 \leq x \leq 9$, $h(1) = 30$ and $h(7) = 45$. Thus the graph of $y = h(x)$ is a straight line, going through $(1, 30)$ and $(7, 45)$. The equation is thus

$$\frac{y-30}{x-1} = \frac{45-30}{7-1} = \frac{5}{2};$$

that is, $h(x) = 30 + \frac{5}{2}(x-1)$, so that
a. $h(9) = 30 + \frac{5}{2}(9-1) = 50$ inches; and for

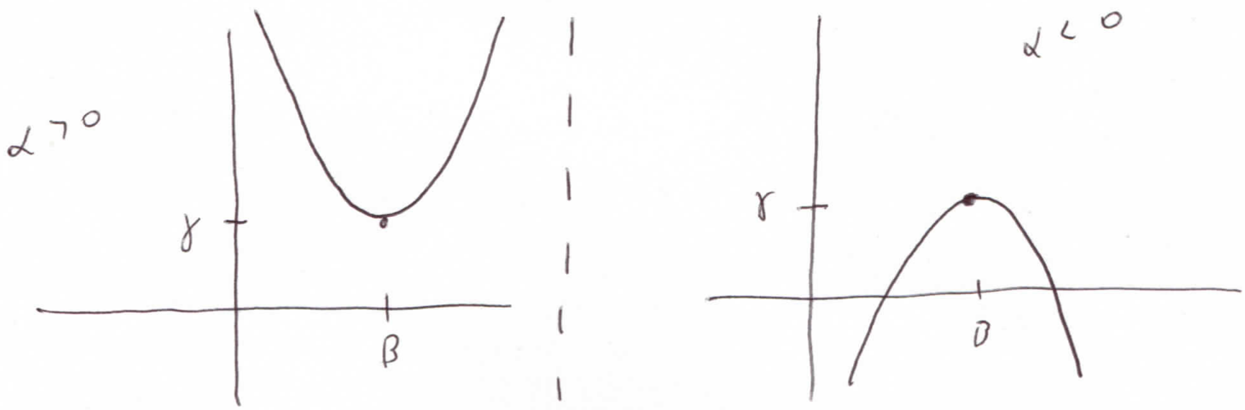
b. set $40 = h(x) = 30 + \frac{5}{2}(x-1)$, and solve for x: $10 = \frac{5}{2}(x-1) \rightarrow (x-1) = \left(\frac{2}{5}\right) 10 = 4 \rightarrow x = 5$ years old. ∎

2.20. A second degree polynomial is a **quadratic function**; its graph is called a **parabola**. When a parabola is rewritten as

$$y = \alpha(x-\beta)^2 + \gamma, \tag{2.21}$$

(see 1.29) then we may immediately make a rough graph. The point (β, γ) is the **vertex**.

The graph will look like one of the following two possibilities. Note how, in both cases, it's relatively flat near the vertex, then rises (if $\alpha > 0$) or falls (if $\alpha < 0$) more quickly as you get further away from the vertex.

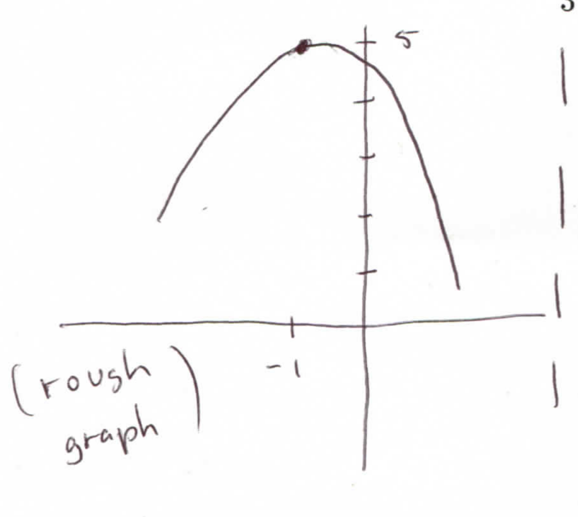


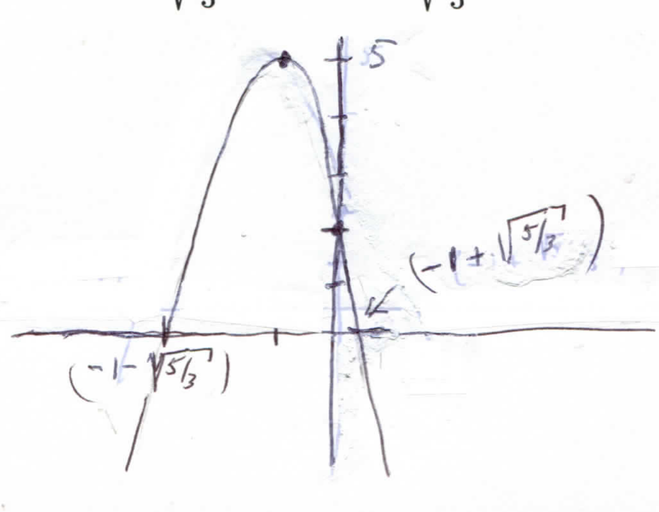
Example 2.22. Graph $y = f(x)$, and find the vertex, if

$$f(x) = -3(x+1)^2 + 5.$$

Solution. Here $\alpha = -3, \beta = -1, \gamma = 5$ (from (2.21)), so the vertex is $(-1, 5)$. Since α is negative, the graph goes down from its vertex. This is all we need for a rough graph (on the left below). To refine our drawing, let's throw in intercepts. The y-intercept is $f(0) = 2$. For the x-intercept, set $f(x) = 0$ and solve for x:

$$0 = -3(x+1)^2 + 5 \rightarrow (x+1)^2 = \frac{5}{3} \rightarrow (x+1) = \pm\sqrt{\frac{5}{3}} \rightarrow x = -1 \pm \sqrt{\frac{5}{3}}.$$





2.23. How to rewrite a quadratic function to get vertex and graph: complete the square.

Examples 2.24. For each of the following, graph $y = f(x)$, and find the vertex and intercepts.

a. $f(x) = x^2 + 6x + 10$;

b. $f(x) = 8x - 2x^2 + 3$

Solutions.

a. We are graphing $y = f(x) = x^2 + 6x + 10$. Complete the square, as in 1.29, Steps (1)–(5):

$$y - 10 = x^2 + 6x \quad \rightarrow \quad y - 10 + 3^2 = (x^2 + 6x + 3^2) = (x + 3)^2,$$

so

$$y = (x + 3)^2 + 1;$$

thus the vertex is $(-3, 1)$ and the graph goes up, since α, in (2.21), equals one.

The y intercept is $f(0) = 10$. For the x intercepts, set $y = 0$ and (try to) solve for x:

$$0 = (x + 3)^2 + 1 \quad \rightarrow \quad (x + 3)^2 = -1 \quad \rightarrow \quad (x + 3) = \pm\sqrt{-1}??$$

Since $\sqrt{-1}$ is not real, this is saying that the graph has no x intercepts.

b. We are graphing $y = f(x) = -2x^2 + 8x + 3$, so we complete the square as in 1.29:

$$\frac{y - 3}{-2} = (x^2 - 4x) \quad \rightarrow \quad \frac{y - 3}{-2} + (-2)^2 = (x^2 - 4x + (-2)^2) = (x - 2)^2,$$

thus, solving for y gives us

$$y = -2(x - 2)^2 + 11,$$

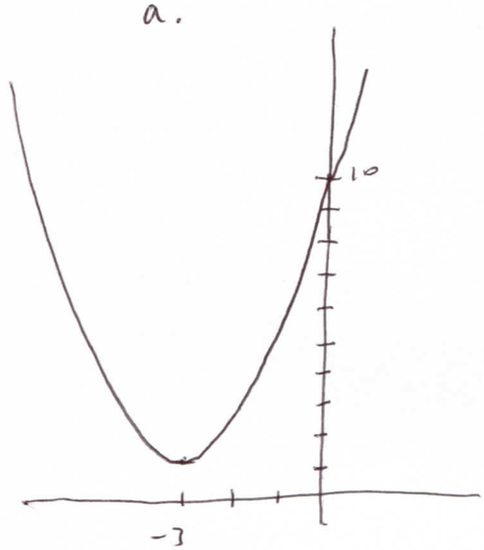
so that the vertex is $(2, 11)$. The graph goes down, since $\alpha = -2$. (see (2.21))

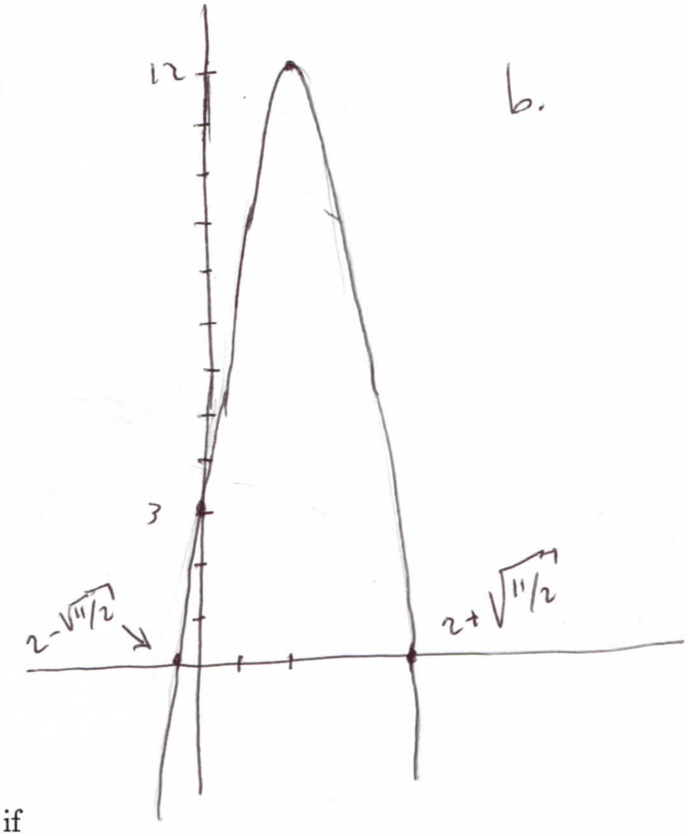
The y intercept is $f(0) = 3$. For the x intercepts, we try to solve for x when $y = 0$:

$$0 = -2(x - 2)^2 + 11 \quad \rightarrow \quad (x - 2)^2 = \frac{11}{2} \quad \rightarrow \quad x = 2 \pm \sqrt{\frac{11}{2}}$$

are the x intercepts.

Here are the graphs.



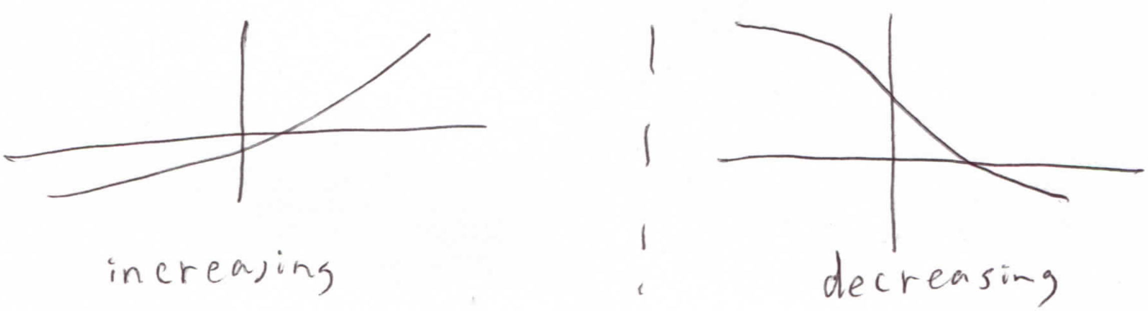


2.25. A function f is **increasing** on (a, b) if

$$f(x_1) > f(x_2) \quad \text{whenever} \quad x_1 > x_2,\ x_1, x_2 \text{ in } (a, b).$$

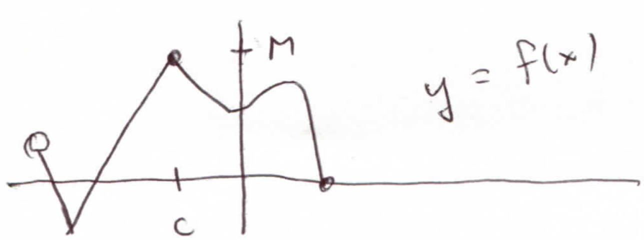
f is **decreasing** on (a, b) if

$$f(x_1) < f(x_2) \quad \text{whenever} \quad x_1 > x_2,\ x_1, x_2 \text{ in } (a, b).$$



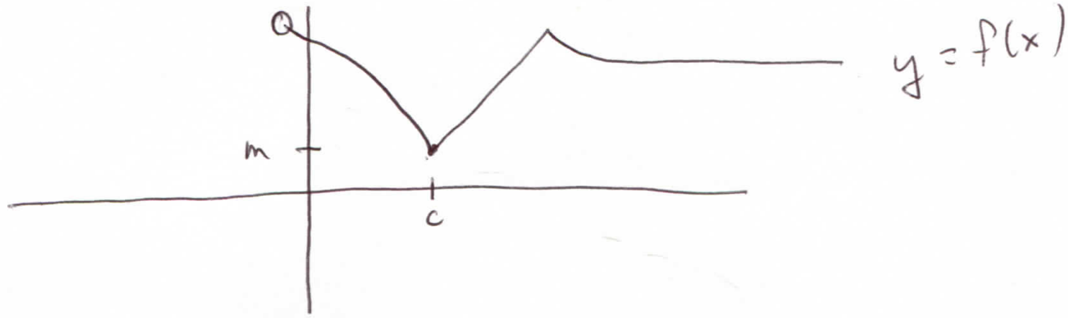
2.26. The function f has a **maximum value** of M, occurring at c, if

$$M = f(c) \geq f(x) \quad \text{for all } x \text{ in the domain of } f.$$



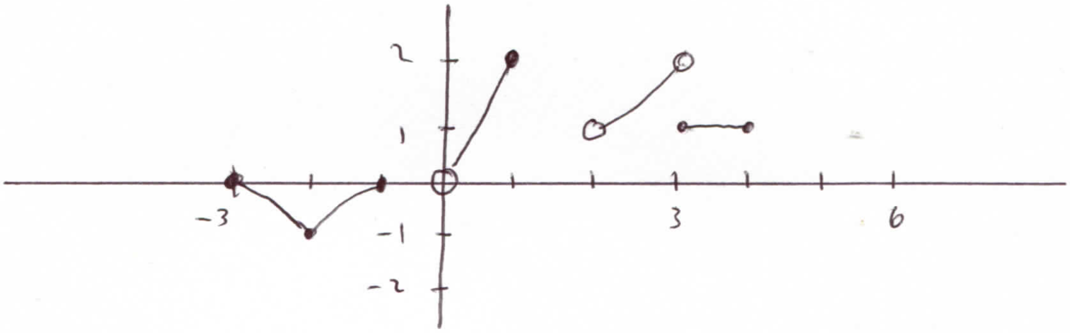
f has a **minimum value** of m, occurring at c, if

$$m = f(c) \leq f(x) \quad \text{for all } x \text{ in the domain of } f.$$



You must distinguish between the maximum value M, where the maximum value occurs (at c), and the point on the graph (c, M). If a question asks "What is the maximum value of f?", your answer would be M. If a question asks "at what number does the maximum value occur?", your answer would be c.

Example 2.27. Use the graph below to find the domain and range of the function f, where it's increasing and decreasing, where the maximum or minimum occurs, and what the maximum or minimum is.



Solution. The domain is $\{x \mid 3 \leq x \leq -1, 0 < x \leq 1 \text{ or } 2 < x \leq 4\}$. The range is $[-1, 2]$. The maximum value is 2, occurring at $x = 1$. The minimum value is -1, occurring at $x = -2$. The graph is increasing on $(-2, -1), (0, 1)$ and $(2, 3)$. The graph is decreasing on $(-3, -2)$.

For a linear function, the slope tells you whether it's increasing or decreasing: positive slope means it's increasing everywhere, negative slope means it's decreasing everywhere. A nonconstant linear function has no maximum or minimum on the real line.

For quadratic functions, we may get the maximum or minimum value explicitly.

2.28. Maximum/minimum of quadratic functions: If

$$f(x) = \alpha(x - \beta)^2 + \gamma,$$

as in (2.21), then γ is its maximum value (if $\alpha < 0$) or its minimum value (if $\alpha > 0$). The maximum or minimum value occurs at $x = \beta$.

Recall that (β, γ) is the vertex of the graph of f (see (2.21)).

Examples 2.29. For each of the following functions, find the maximum or minimum, and find on which intervals the graph is increasing or decreasing.

a. $f(x) = 2x^2 - 12x + 1$.

b. $f(x) = -x^2 - 4x$.

Solutions. The surest way to do this is to draw the graph, as in Examples 2.26, by letting $y = f(x)$ and completing the square as in 1.29, to get the vertex. You could also do this without the graph, just by using 2.28.

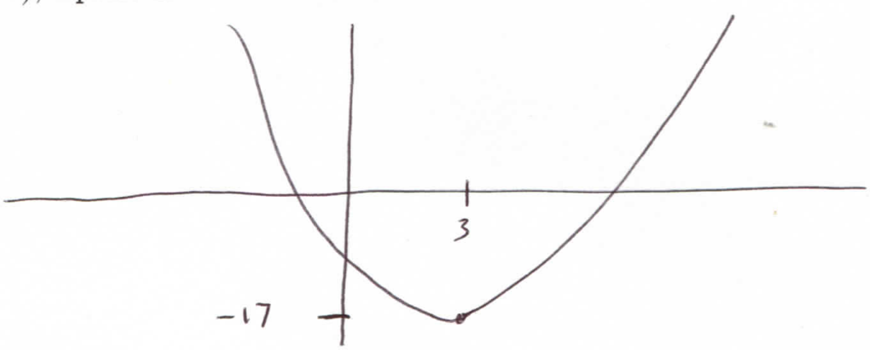
a. Let $y = 2x^2 - 12x + 1$, so that

$$\frac{y-1}{2} = x^2 - 6x \rightarrow \frac{y-1}{2} + (-3)^2 = x^2 - 6x + (-3)^2 = (x-3)^2;$$

solving for y gives us

$$y = 2(x-3)^2 - 17.$$

Thus the graph has a vertex of $(3, -17)$, and, by 2.20, goes up from its vertex, since α, from (2.21), equals 2.



Staring at this rough graph tells us the function is decreasing on $(-\infty, 3)$, increasing on $(3, \infty)$, has a minimum value of -17 and no maximum value.

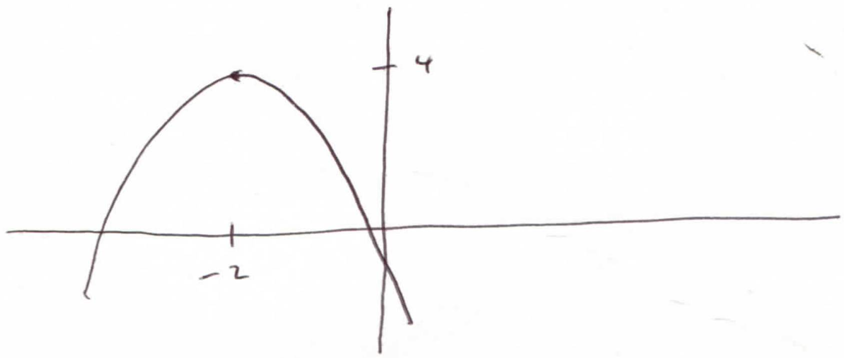
b. Let $y = -x^2 - 4x$, then $-y = x^2 + 4x$, so that

$$-y + 2^2 = x^2 + 4x + 2^2 = (x+2)^2,$$

or

$$y = -(x+2)^2 + 4.$$

Thus we have a vertex of $(-2, 4)$, and the graph goes down from its vertex, since α, from (2.21), equals -1.

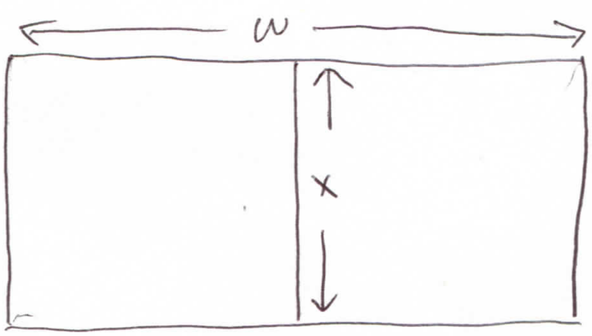


The graph tells us that the function is increasing on $(-\infty, -2)$, decreasing on $(-2, \infty)$, has a maximum value of 4 and no minimum value. ∎

Example 2.30. Suppose I have 300 dollars to spend on fencing. I want a rectangle fenced off, and cut in half with an interior fence. The interior fencing costs one dollar per foot, and the exterior fencing costs two dollars per foot.

Find the dimensions of the fenced-off rectangle of maximum area.

Solution. Let x be the length, in feet, of the interior fence, and let w (for width) be the other dimension, in feet, of the fenced-off rectangle.



(*)

We want to maximize the area xw. Before we can do anything, we must get w in terms of x, so that we will have a function purely of x that we can maximize. We can relate x and w by using the cost restriction in the first sentence of the problem. We have x feet of interior fence, at one dollar per foot; that's x dollars spent. Then we have $(2x + 2w)$ feet of exterior fence (see (*)), at two dollars per foot; that's $2(2x + 2w)$ dollars spent. Since we're spending 300 dollars altogether,

$$300 = x + 2(2x + 2w).$$

Solving for w gives us

$$w = 75 - \frac{5}{4}x. \qquad (**)$$

Now we can write down the area function that we want to maximize:

$$f(x) = xw = x(75 - \frac{5}{4}x).$$

2.28 tells us to complete the square. For convenience and familiarity, let's let $y = f(x) = x(75 - \frac{5}{4}x) = -\frac{5}{4}x^2 + 75x$, so that

$$-\frac{4}{5}y = x^2 - 60x \quad \to \quad -\frac{4}{5}y + 30^2 = x^2 - 60x + 30^2 = (x - 30)^2;$$

solving for y gives us

$$f(x) = y = -\frac{5}{4}(x - 30)^2 + \left(\frac{5}{4}\right)(900).$$

2.28 tells us that $f(x)$ has a maximum value of $(\frac{5}{4})(900)$, occurring at $x = 30$.

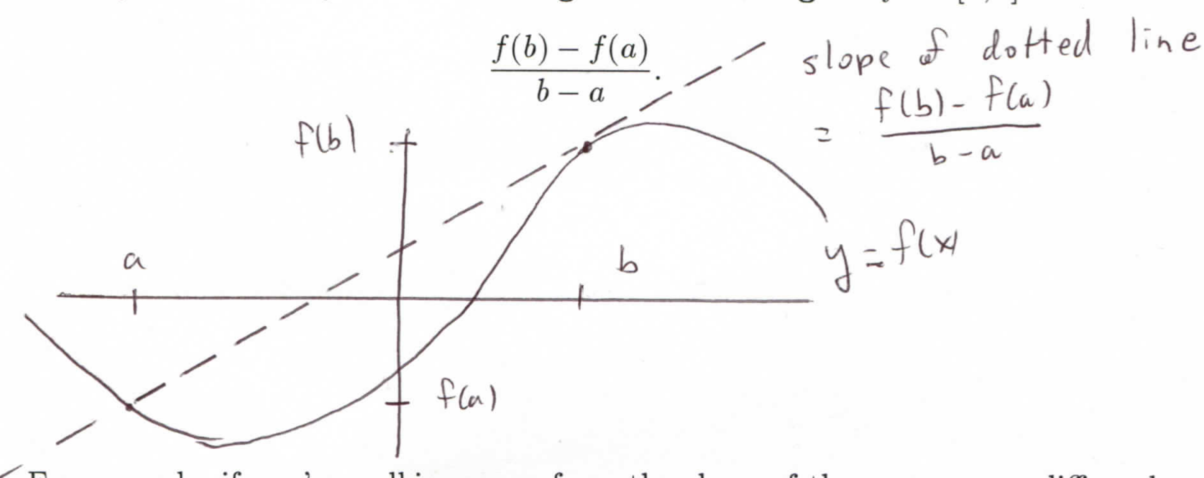
The question is asking for the dimensions that give the maximum area; that means we want x and w. We've got x, so we can now solve for w (see (**)):

$$w = 75 - \frac{5}{4}x = 75 - \frac{5}{4}(30) = \frac{75}{2}.$$

Our desired dimensions are $x = 30$ feet, $w = \frac{75}{2}$ feet (see (*)). ∎

2.31. If f is a function, then the **average rate of change** of f on $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}.$$



For example, if you're walking away from the shore of the ocean up a cliff, and $f(x)$ is your height, in feet, above sea level x feet away from the shore, then the average rate of change of f measures how steep the cliff is.

This leads to the biggest deal in calculus, the "derivative," or instantaneous rate of change. But not in this class.

You should be able to show that the average rate of change of a linear function, on any interval, is the slope of its graph.

Examples 2.32. a. Find the average rate of change of $f(x) = 3x^2 + x - 19$ on $[1, 3]$, and simplify.

b. Same as a, on $[a, b]$, for a, b arbitrary.

c. Find the average rate of change of $f(x) = \sqrt{x}$ on $[1, 4]$ and simplify.

d. Same as c, on $[a, b]$, for a, b arbitrary.

e. Find the average rate of change of $f(x) = \frac{1}{x}$ on $[.1, .5]$, and simplify.

Solutions. a. $\frac{f(3)-f(1)}{3-1} = \frac{11-(-15)}{3-1} = 13.$

b.

$$\frac{f(b) - f(a)}{b - a} = \frac{(3b^2 + b - 19) - (3a^2 + a - 19)}{b - a} = \frac{3(b^2 - a^2) + (b - a)}{b - a}$$
$$= \frac{3(b + a)(b - a) + (b - a)}{b - a} = 3(b + a) + 1.$$

c. $\frac{\sqrt{4}-\sqrt{1}}{4-1} = \frac{2-1}{4-1} = \frac{1}{3}.$

d. The following technique is called "multiplying by the conjugate"; $(\sqrt{b} + \sqrt{a})$ is the conjugate of $(\sqrt{b} - \sqrt{a})$. The reason it works is the difference of two squares formula 1.23a.

$$\frac{\sqrt{b} - \sqrt{a}}{b - a} = \frac{(\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a})}{(b - a)(\sqrt{b} + \sqrt{a})} = \frac{(b - a)}{(b - a)(\sqrt{b} + \sqrt{a})} = \frac{1}{(\sqrt{b} + \sqrt{a})}.$$

e.

$$\frac{f(.5) - f(.1)}{.5 - .1} = \frac{\frac{1}{.5} - \frac{1}{.1}}{.5 - .1} = \frac{2 - 10}{.4} = -20.$$

2.33. If $f(t)$ is your position after t minutes, then your **average velocity** over t in $[a, b]$ is the average rate of change of f on $[a, b]$.

Example 2.34. Suppose that you are driving away from Athens. At 1 AM you are ten miles from Athens, and at 3 AM you are seventy miles from Athens. What is your average velocity over the time between 1 AM and 3 AM?

Solution. Let $f(t)$ be the distance from Athens t hours after midnight. We want

$$\frac{f(3)-f(1)}{3-1} = \frac{70-10}{3-1} = 20 \text{ miles per hour} .$$

Here's something else whose instantaneous analogue will be a big deal in calculus.

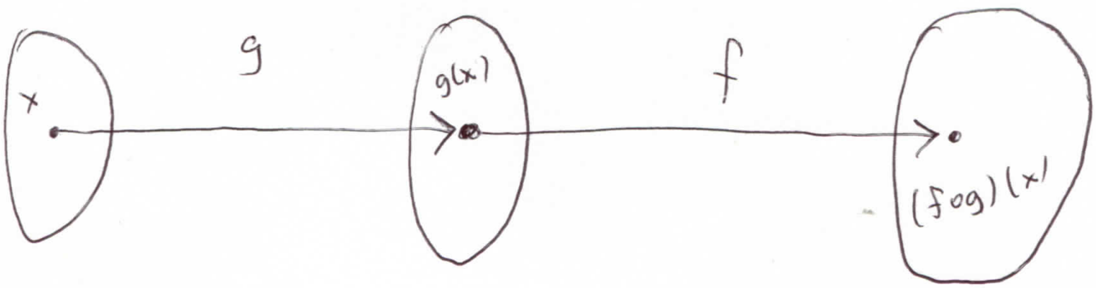
Proposition 2.35. *If the average rate of change of f on $[x,y]$ is positive, for all $y > x$ in $[a,b]$, then f is increasing in $[a,b]$.*

2.36. If f and g are two functions, then $f \circ g$, the **composition** of f with g, is defined by

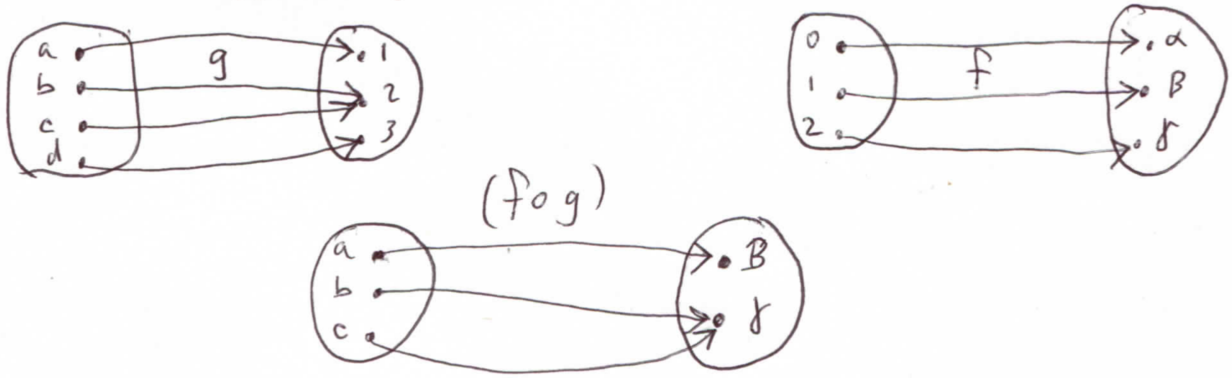
$$(f \circ g)(x) \equiv f(g(x))$$

for all x in the domain of $f \circ g$, defined to be the set of all x in the domain of g such that $g(x)$ is in the domain of f.

Here's what happens to a single point.



Here's what happens to a pair of functions.



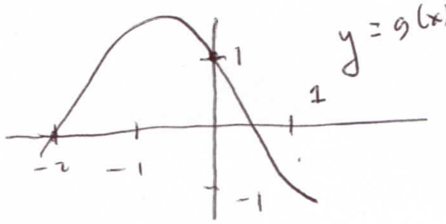
Examples 2.37. a. Find $(f \circ g)$, $(f \circ f)$ and $(g \circ f)$ if $f(x) = 3x + 2$ and $g(x) = x^2$.
b. Find $(f \circ g)$ and $(g \circ f)$, if $g(x) = -x^2$ and $f(x) = \sqrt{x}$.
c. Write $h(x) \equiv \sqrt{x^2+1}$ as a composition of two simpler functions.
d. Find the graph of $(f \circ g)$ and $(g \circ f)$ if the graph of f is $\{(1,2),(2,-5),(3,0),(4,0)\}$ and the graph of g is $\{(0,1),(1,19),(2,7)\}$.

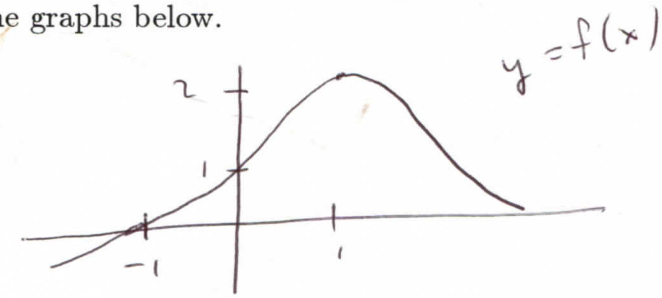
e. Find $(f \circ g)$ and $(g \circ f)$ if f and g are given by the following tables.

x	-1	0	1	2	3	4	5
$f(x)$	0	3	17	1	0	-1	-1

x	0	1	2	3
$g(x)$	-1	5	0	4

f. Estimate $(f \circ g)(1)$ and $(f \circ g)(0)$ from the graphs below.

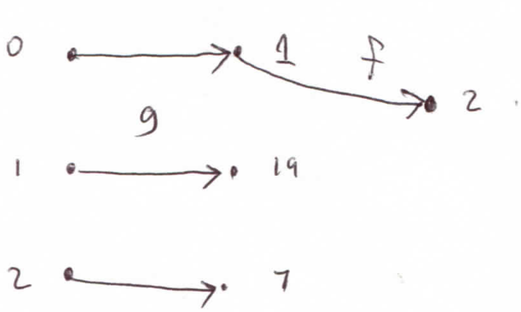


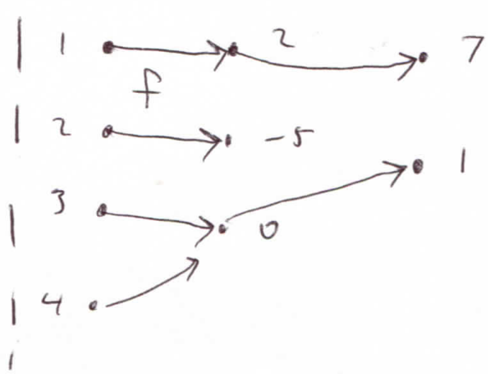


Solutions. a. $(f \circ g)(x) \equiv f(g(x)) = 3x^2 + 2$.
$(g \circ f)(x) \equiv g(f(x)) = (3x+2)^2 = 9x^2 + 12x + 4$.
$(f \circ f)(x) \equiv f(f(x)) = 3(3x+2) + 2 = 9x + 8$.
b. $(f \circ g)(x) \equiv f(g(x)) = \sqrt{-x^2}$. This is defined only when $x = 0$. In fact, if you look at the definition of the domain of $(f \circ g)$ in 2.36, you see that x is in the domain of $(f \circ g)$ if and only if $-x^2$ is in the domain of f. Since the domain of f is $[0, \infty)$, the domain of $(f \circ g)$ is the set of all x such that $-x^2 \geq 0$, or $x^2 \leq 0$. Thus the domain of $(f \circ g)$ is $\{0\}$, and

$$(f \circ g)(0) = \sqrt{-0^2} = 0.$$

The graph of $(f \circ g)$ is $\{(0,0)\}$.
$(g \circ f)(x) \equiv g(f(x)) = -\left(\sqrt{x}\right)^2 = -x$, with domain $[0, \infty)$ (the domain of f).
c. $h = f \circ g$, where $g(x) = x^2 + 1$, $f(x) = \sqrt{x}$. This is not the only correct answer; you could also have $g(x) = x^2$, $f(x) = \sqrt{x+1}$.
d. The graph of $(f \circ g)$ is $\{(0,2)\}$. The graph of $(g \circ f)$ is $\{(1,7),(3,1),(4,1)\}$.





e.

x	0	1	2	3
$(f \circ g)(x)$	0	-1	3	-1

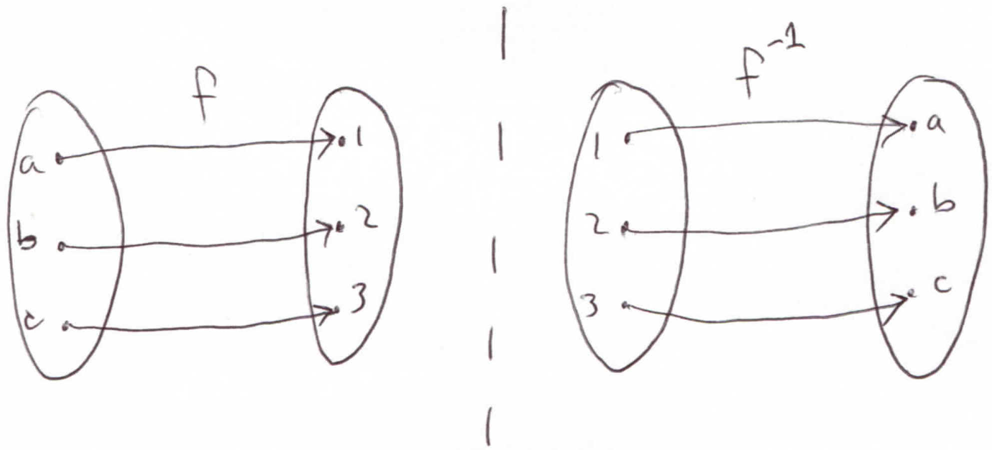
x	-1	0	2	3
$(g \circ f)(x)$	-1	4	5	-1

f. $(f \circ g)(1) \equiv f(g(1)) \sim f(-1) \sim 0$; $(f \circ g)(0) \equiv f(g(0)) \sim f(1) \sim 2$. ∎

2.38. If f and g are functions, with the range of g contained in the domain of f, the range of f contained in the domain of g, and

$$(f \circ g)(x) = x \quad \text{for } x \text{ in domain of } g \text{ and } \quad (g \circ f)(x) = x \quad \text{for } x \text{ in domain of } f,$$

then g is the **inverse function** of f, written $g = f^{-1}$.



The inverse function of f undoes whatever f did. Again thinking of a function as action, f does something to x, to change it into $f(x)$; f^{-1} then does something to $f(x)$ that takes you right back to x: $f^{-1}(f(x)) = x$.

An inverse function is also thinking backwards. Imagine you have a coherent distance function: $d(t)$ equal to miles travelled after t minutes. The inverse function of d is the restless voice of the child in the back seat asking "When are we going to be there?" For example, if you must travel 300 miles to "be there," this child is asking for $d^{-1}(300)$.

Examples 2.39. a. Find f^{-1}, if f is given by the following table.

x	-1	0	2	3	19
$f(x)$	1	2	0	9	7

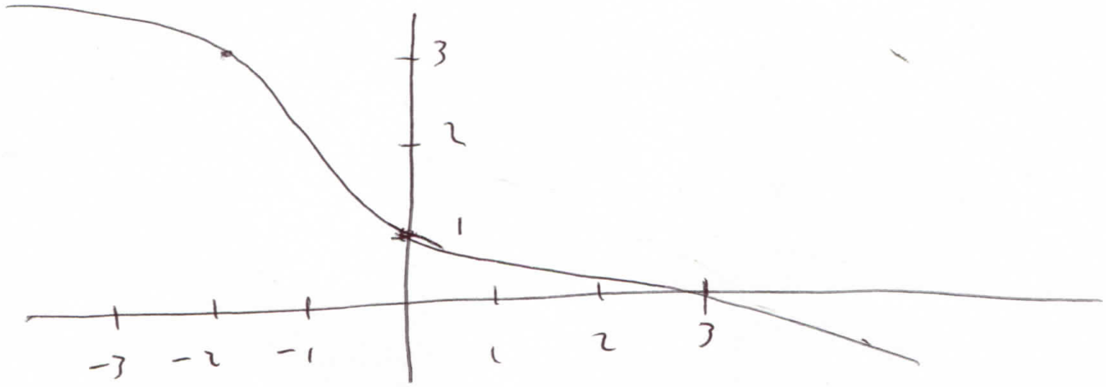
b. Find the graph of f^{-1}, if the graph of f is $\{(0,1),(1,2),(2,-1),(3,3)\}$.

c. If $f(-2) = 5$ and f has an inverse function, what is $f^{-1}(5)$?

d. Find $f^{-1}(4)$, if $f(x) = x^2 + 2x + 1$, with domain of f equal to $[-1, \infty)$.

e. Find $f^{-1}(-2)$, if $f(x) = \frac{3x-4}{2x+1}$.

f. Use the graph of f below to estimate $f^{-1}(3)$.



Solutions. a. Flip the table:

x	1	2	0	9	7
$f^{-1}(x)$	-1	0	2	3	19

b. Flip the ordered pairs: The graph of f^{-1} is $\{(1,0),(2,1),(-1,2),(3,3)\}$.
c. $f^{-1}(5) = f^{-1}(f(2)) = 2$, by 2.38.
d. Temporarily give $f^{-1}(4)$ a name: $x \equiv f^{-1}(4)$. This means that

$$f(x) = f(f^{-1}(4)) = 4,$$

by 2.38; that is,

$$x^2 + 2x + 1 = 4 \rightarrow x^2 + 2x - 3 = 0 \rightarrow (x+3)(x-1) = 0;$$

thus $x = 1$ or -3. Since the domain of f is $[-1, \infty)$, $x \geq -1$. Thus x cannot be -3, so $x = 1$; $f^{-1}(4) = 1$.
e. As in d., let $x \equiv f^{-1}(-2)$, so that

$$\frac{3x-4}{2x+1} = f(x) = f(f^{-1}(-2)) = -2,$$

by 2.38; now solve for x:

$$3x - 4 = -2(2x+1) = -4x - 2 \rightarrow x = \frac{2}{7};$$

$f^{-1}(-2) = \frac{2}{7}$.
f. $f(-2) \sim 3$, so $f^{-1}(3) \sim -2$. ∎

2.40. Calculating f^{-1}: Set $y = f(x)$, and solve for $x = f^{-1}(y)$.
Compare this with 2.12.

Examples 2.41. In each of the following, find f^{-1}.

a. $f(x) = -2x + 4$.

b. $f(x) = 2x^2 - 4x$, domain of f equal to $(-\infty, 1]$.

c. $f(x) = \frac{1-x}{2x+3}$.

Solutions. a. Set $y = -2x + 4$ and solve for x: $-2x = y - 4 \to x = 2 - \frac{y}{2}$. So $f^{-1}(y) = 2 - \frac{y}{2}$ is the inverse function. If you prefer x for the domain, you could equivalently write this as $f^{-1}(x) = 2 - \frac{x}{2}$.

b. Set $y = 2x^2 - 4x$ and solve for x (use the quadratic formula (1.33)):

$$2x^2 - 4x - y = 0 \;\to\; x = \frac{1}{4}\left[4 \pm \sqrt{(-4)^2 - 4(2)(-y)}\right] = 1 \pm \frac{1}{4}\sqrt{16 + 8y}.$$

This doesn't even look like a function, because of the $\pm$. That's why I restricted the domain of f to $(-\infty, 1]$; this is saying that

$$1 \pm \frac{1}{4}\sqrt{16 + 8y} = x \leq 1,$$

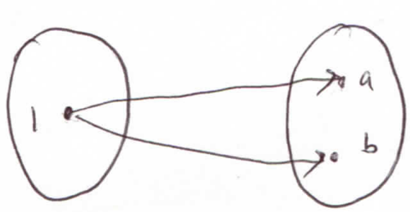
thus we must choose the minus in the $\pm$, so that our inverse function is given by

$$f^{-1}(y) = 1 - \sqrt{16 + 8y}.$$

c. Setting $y = \frac{1-x}{2x+3}$ gives us $(2x+3)y = 1 - x$ so that $1 - 3y = 2xy + x = x(2y+1)$, and $f^{-1}(y) = x = \frac{1-3y}{2y+1}$.

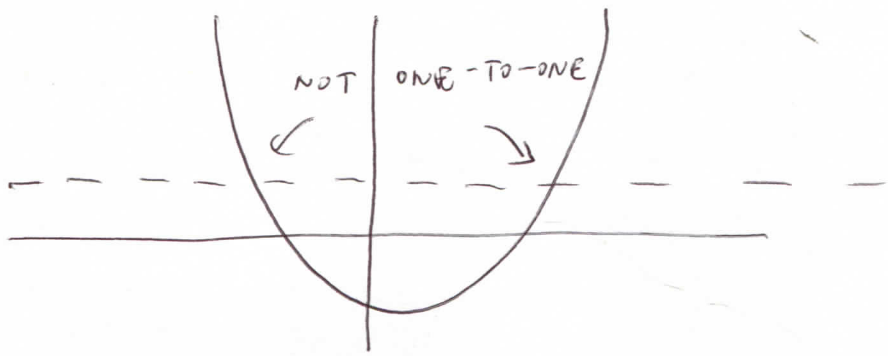
2.42. A function is **one-to-one** if $f(x_1) = f(x_2)$ only when $x_1 = x_2$. This behaviour can be identified from the graph. A function is one-to-one if and only if it satisfies the **"horizontal line test"**: any horizontal line crosses the graph at most once (compare with the vertical line test 2.7).

Note what would happen if we tried to turn around the second drawing to get f^{-1}:



The f^{-1} that I drew is not a function. Being one-to-one is needed so that the inverse is a function.

Here is an illustration of the horizontal line test, for using the graph of f to determine if it is one-to-one.



Theorem 2.43. *f has an inverse function if and only if f is one-to-one.*

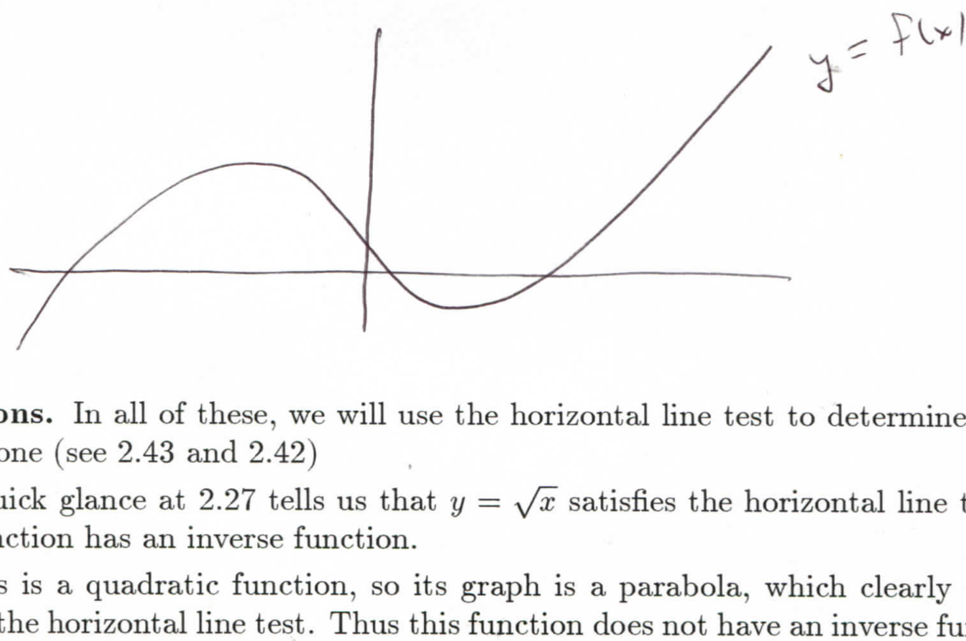
Examples 2.44. Determine, without calculating an inverse, which of the following functions have an inverse function.

a. $f(x) = \sqrt{x}$.

b. $f(x) = x^2 - 4x + 9$.

c. $f(x) = x^2 + 2x + 4$, domain of f equal to $[0, \infty)$.

d. f has the following graph.



Solutions. In all of these, we will use the horizontal line test to determine if f is one-to-one (see 2.43 and 2.42)

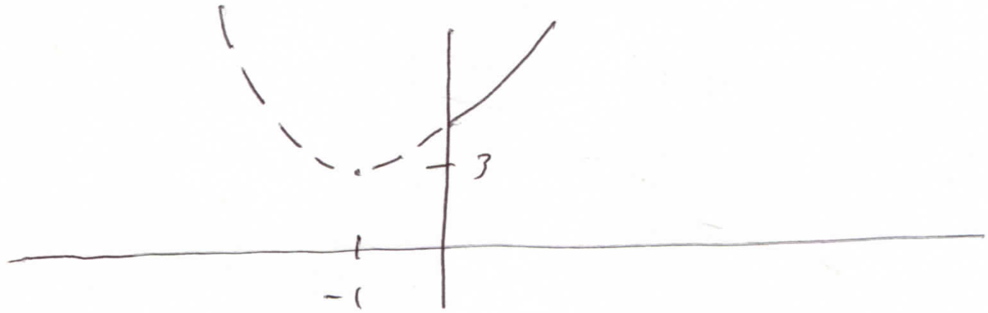
a. A quick glance at 2.27 tells us that $y = \sqrt{x}$ satisfies the horizontal line test, so this function has an inverse function.

b. This is a quadratic function, so its graph is a parabola, which clearly cannot satisfy the horizontal line test. Thus this function does not have an inverse function.

c. The graph of this function is another parabola, except that part of it got chopped off by restricting the domain to $[0, \infty)$. We need to see where the vertex of the parabola $y = x^2 + 2x + 4$ is, by completing the square (see 2.20):

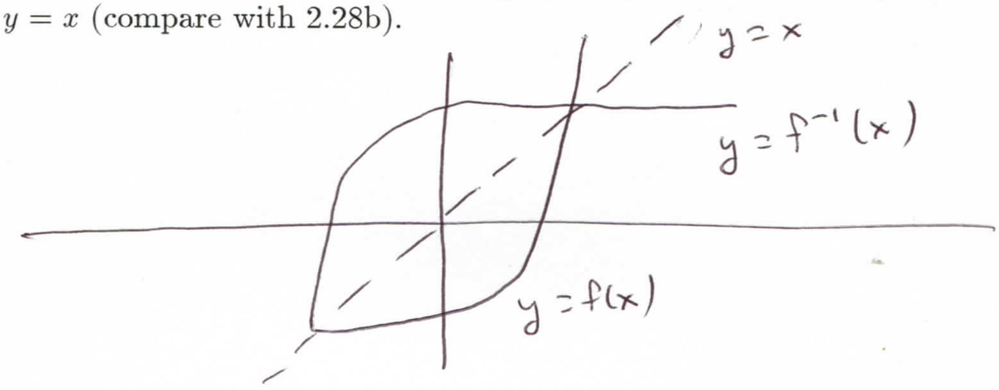
$$y - 4 + 1 = x^2 + 2x + 1 = (x+1)^2 \quad \rightarrow \quad y = (x+1)^2 + 3.$$

Thus the vertex of that parabola is $(-1, 3)$, so that the domain of our function is entirely contained on one side of the vertex; this tells us that its graph satisfies the horizontal line test, so that f has an inverse function (see rough graph below; note that only the solid line is the graph of our function).

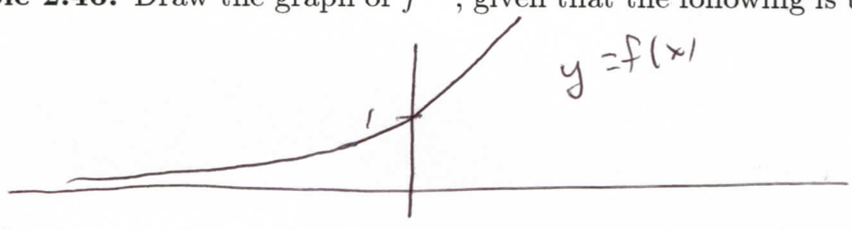


d. The horizontal line test says that f is not one-to-one, therefore does not have an inverse function.

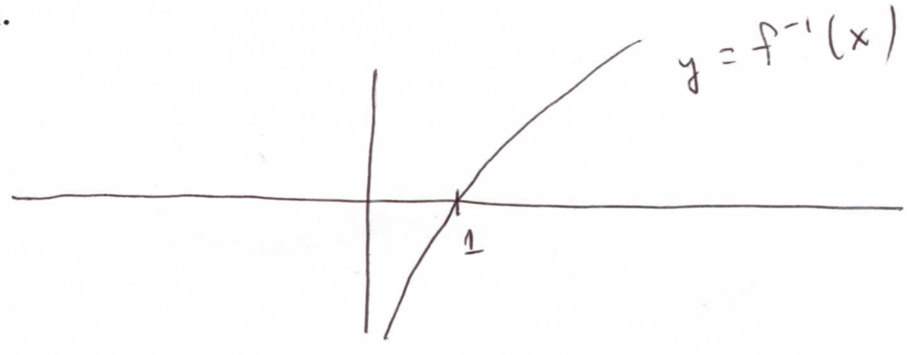
2.45. Graph of f^{-1}: Reflect the graph of f through the line $y = x$; that is, the graph of f^{-1} is the mirror image of the graph of f, if the mirror is placed along the line $y = x$ (compare with 2.28b).



Example 2.46. Draw the graph of f^{-1}, given that the following is the graph of f.

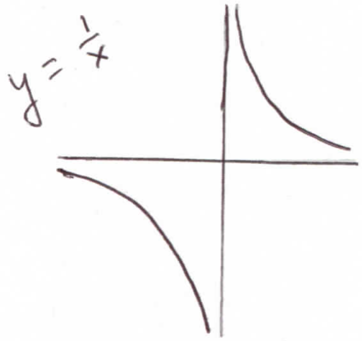


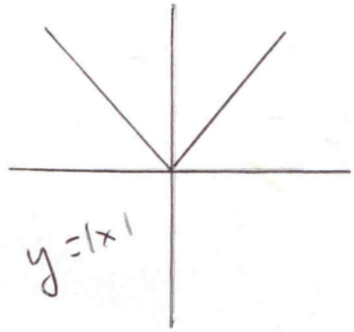
Solution.

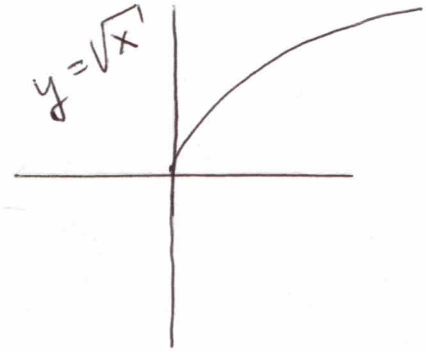


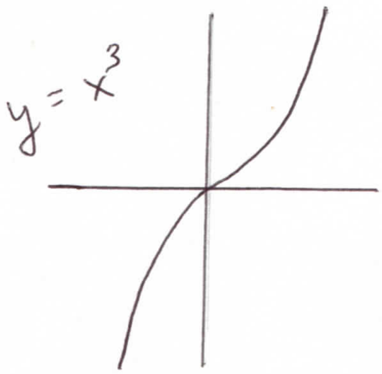
III. MORE GRAPHING

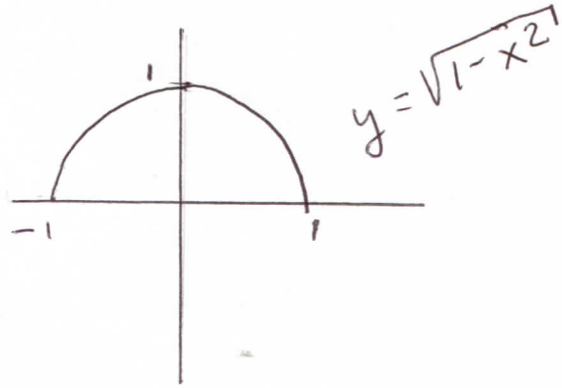
3.1. Other famous graphs. You should memorize the graphs of the following functions: $|x|$, $\frac{1}{x}$, x^3, $\sqrt{x}$, $\sqrt{1-x^2}$





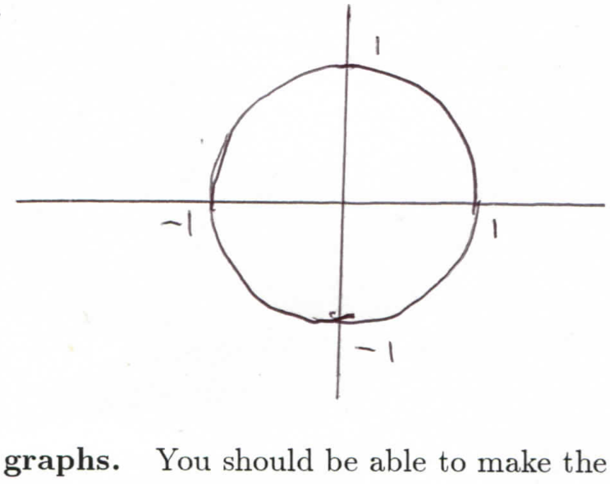






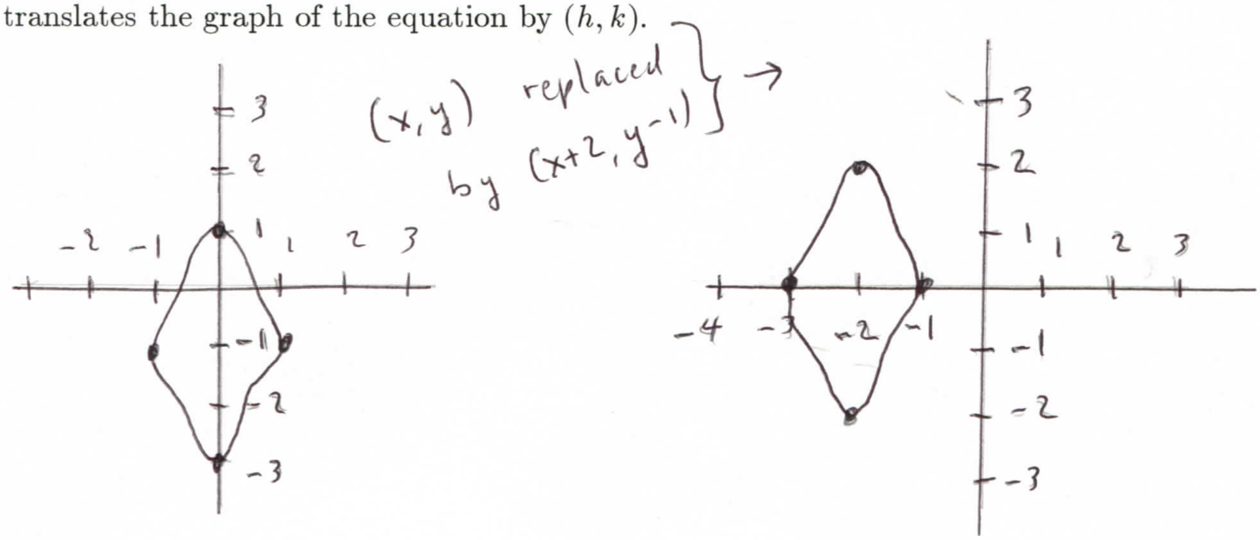
3.2. There are other graphs besides graphs of functions. Given an equation in x and y, the **graph of the equation** is the set of all ordered pairs (x, y) that satisfy the equation.

For example, the graph of $x^2 + y^2 = 1$ is the graph of the circle of radius one centered at the origin.



3.3. Distortions of graphs. You should be able to make the following modifications of the graphs in 3.1, or any other given graph.

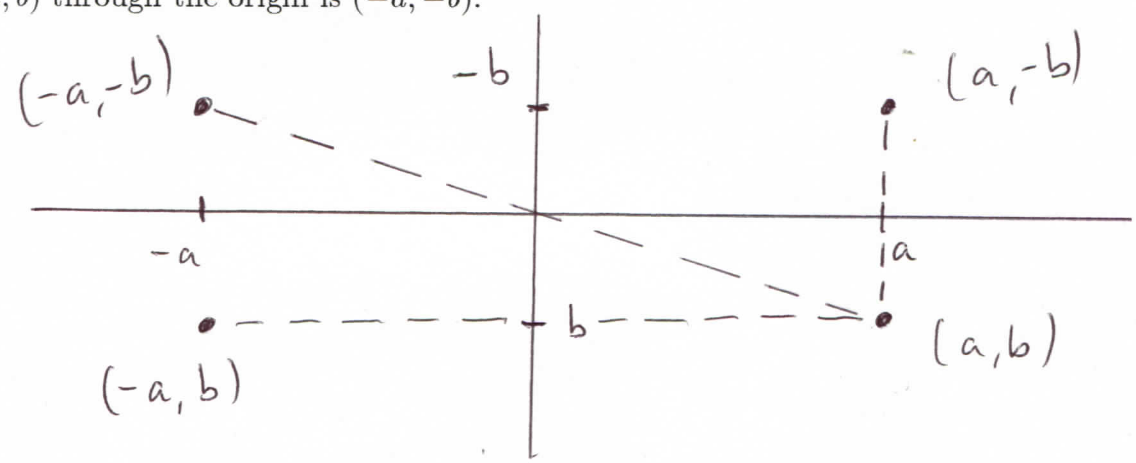
a. **Translation.** Replacing y with $(y-k)$ and x with $(x-h)$ in an equation translates the graph of the equation by (h,k).



We have seen a special case of this kind of translation with parabolas (2.20).

The replacment of y with $(y-k)$ is a **vertical translation**, moving the graph k units up. The replacement of x with $(x-h)$ is a **horizontal translation**, moving the graph h units to the right.

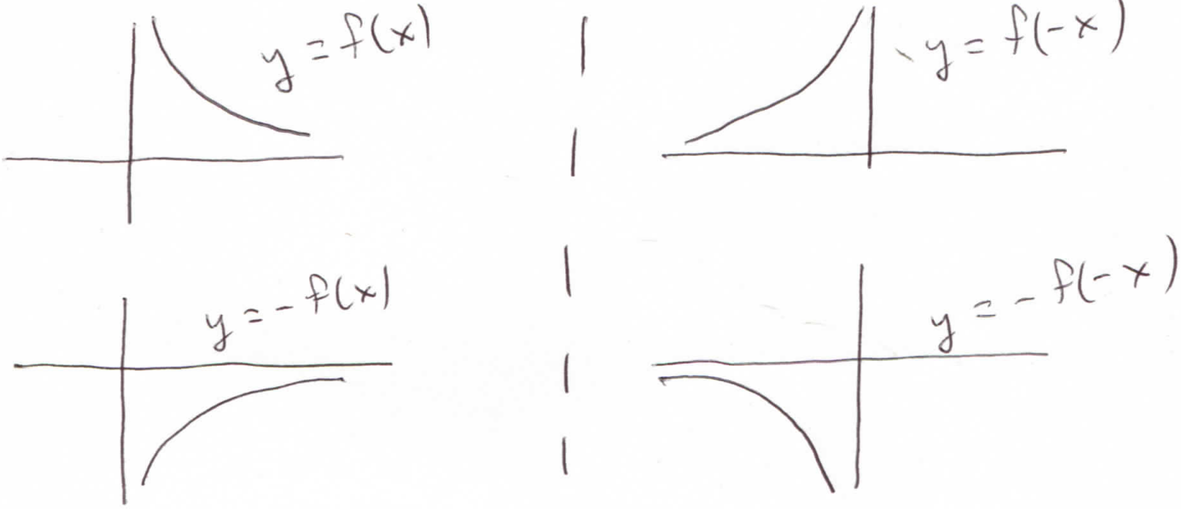
b. **Reflection.** The reflection of the point (a,b) through the x axis is $(a,-b)$. The reflection of the point (a,b) through the y axis is $(-a,b)$. The reflection of the point (a,b) through the origin is $(-a,-b)$.



Reflecting a graph means that you reflect each point on the graph. The reflection of $y=f(x)$ through the x axis is the graph of $y=-f(x)$. The reflection of $y=f(x)$ through the y axis is the graph of $y=f(-x)$. The reflection of $y=f(x)$ through the origin is the graph of $y=-f(-x)$.

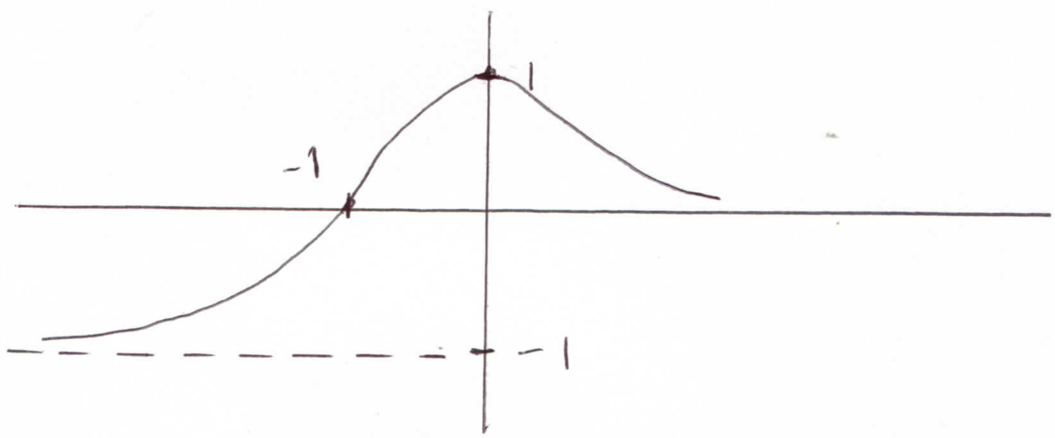
Note that reflection of the graph of a function through the y axis or the origin could change the domain of the function being graphed.

"Reflection" refers to a mirror. The reflection of $y=f(x)$ through the x axis means the mirror image of the graph of f, with the mirror placed on the x axis. The reflection of $y=f(x)$ through the y axis means the mirror image of the graph of f, with the mirror placed on the y axis. Reflection through the origin is reflection through the x axis followed by reflection through the y axis.



c. **Cutting and pasting.** On different intervals, you could have different choices of the famous functions in 3.1.

Examples 3.4. a. Suppose $y = f(x)$ has the following graph. Graph $y = f(x+2)$, $y = 1 + f(x)$, $y = f(-x)$, $y = -f(x)$ and $y = -f(-x)$.



b. Graph $y = 3 + \sqrt{1 - x^2}$.

c. Graph $y = \sqrt{-x}$.

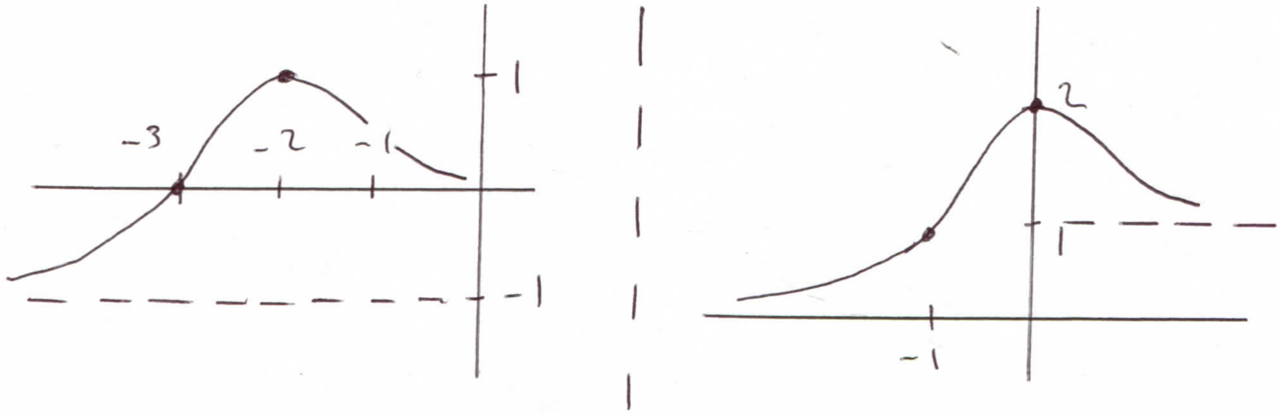
d. Graph $y = 1 - x^3$.

e. Graph $y = \sqrt{x - 2} - 1$.

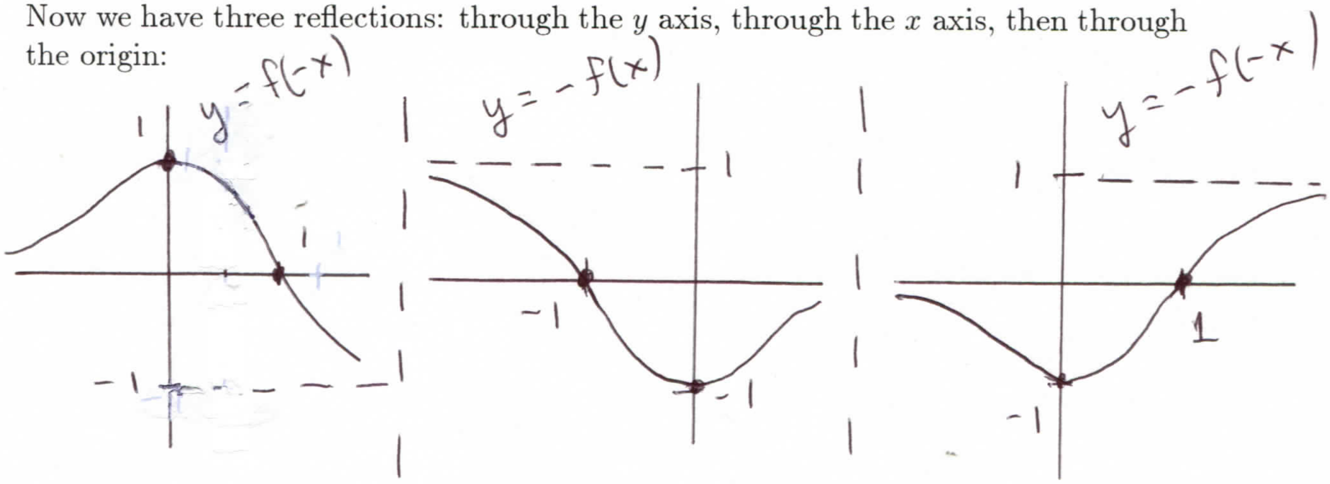
f. Graph

$$f(x) = \begin{cases} \frac{1}{x} & x < -1 \\ |x| & -1 < x \leq 2 \\ \sqrt{x} & 2 < x < 4 \end{cases}$$

Solutions. a. The first graph requested is a horizontal translation 2 units to the left; the second is a vertical translation one unit up:

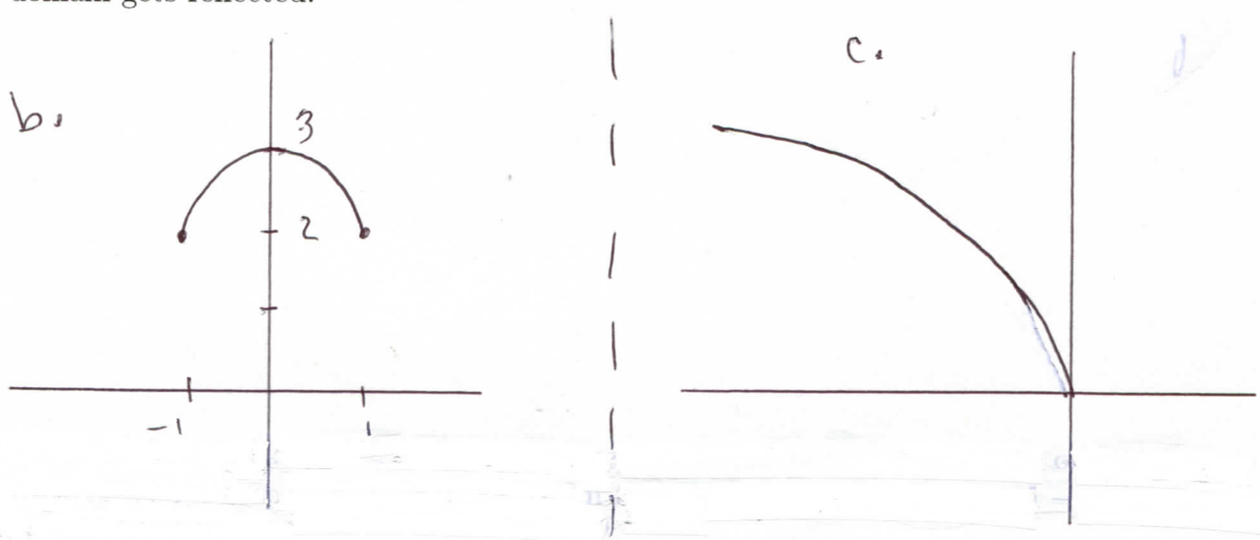


Now we have three reflections: through the y axis, through the x axis, then through the origin:

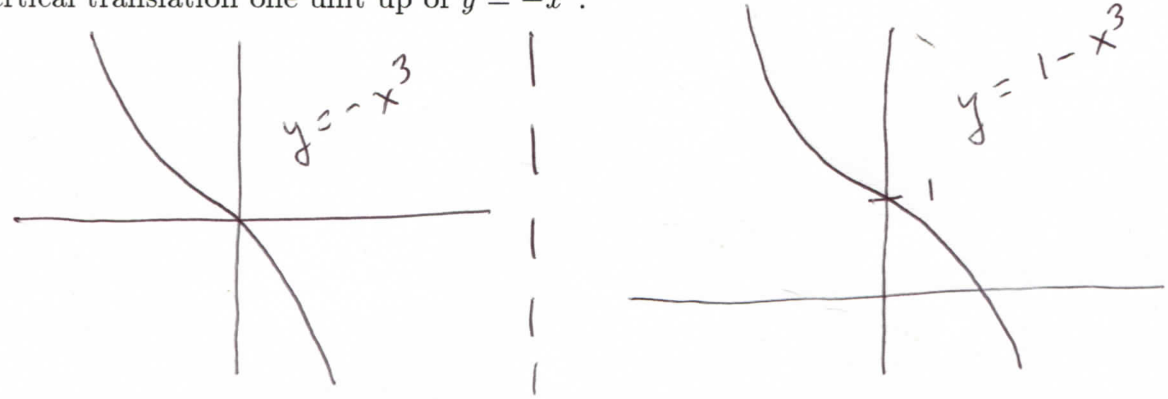


b. This is a vertical translation 3 units up, of $y = \sqrt{1-x^2}$ (see 3.1). I'll put its graph after c.

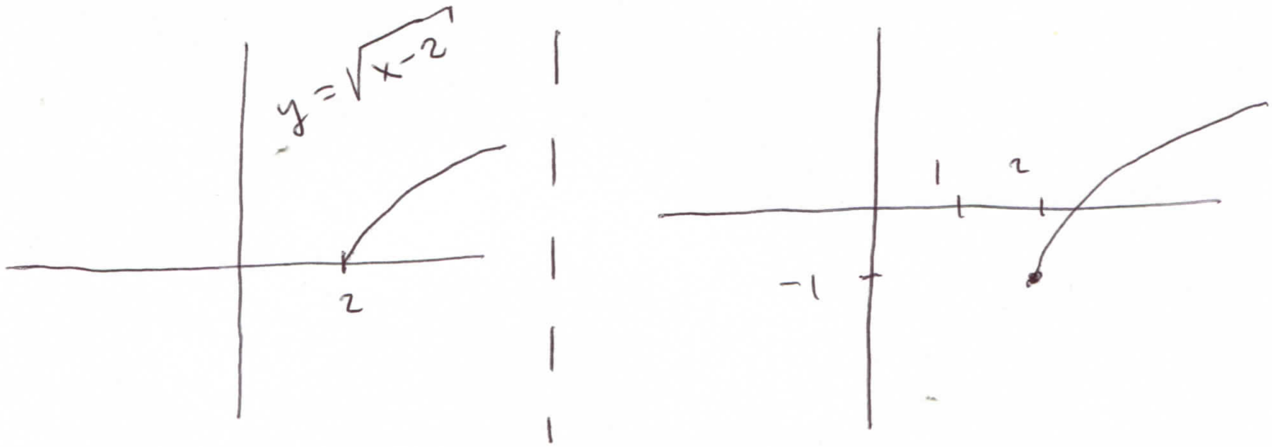
c. This is a reflection of $y = \sqrt{x}$ (see 3.1) through the y axis. Notice that the domain gets reflected.



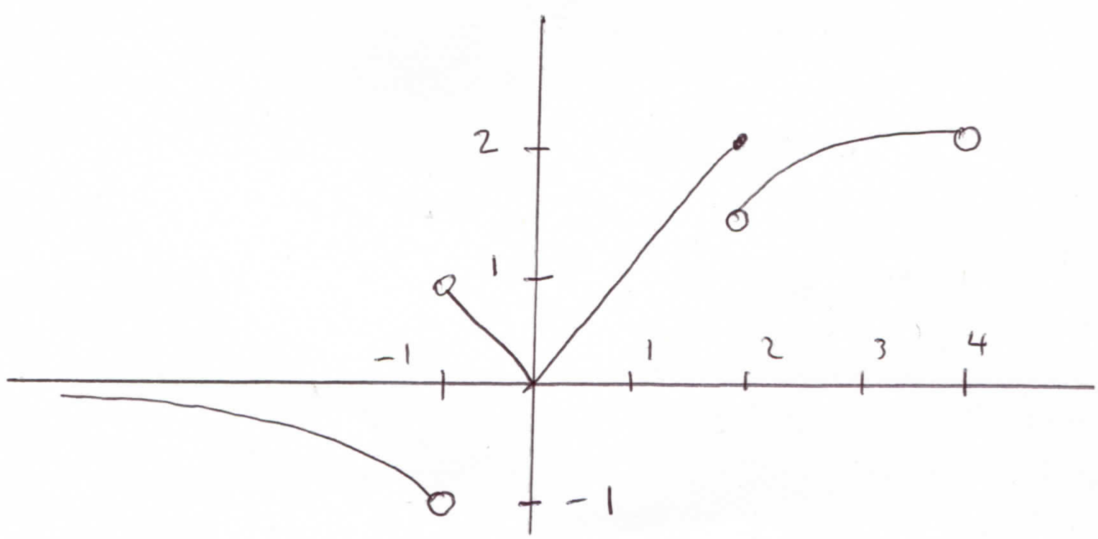
d. Two distortions of $y = x^3$ (see 3.1) are happening here; I'll do them one at a time. First I'll graph $y = -x^3$, a reflection through the x axis; then $y = 1 - x^3$ is a vertical translation one unit up of $y = -x^3$.



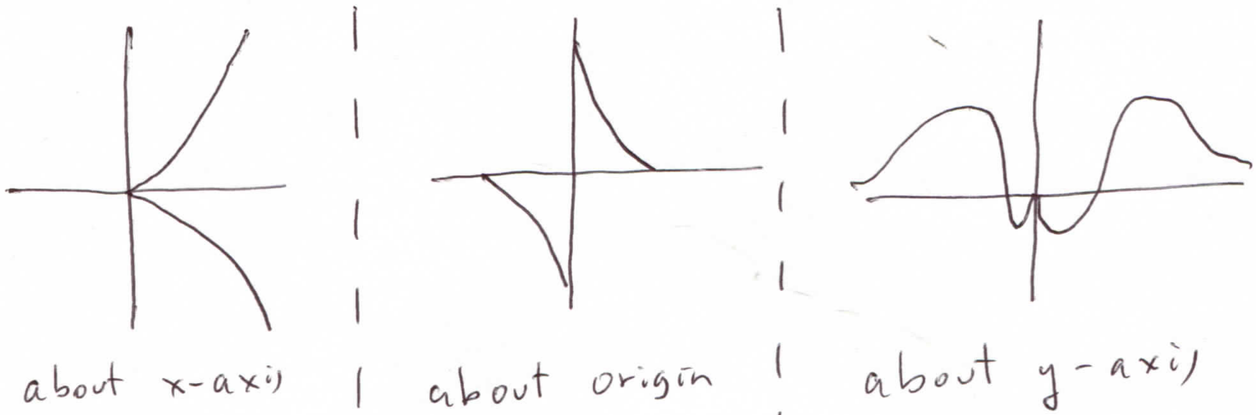
e. This is a translation of $y = \sqrt{x}$ (see 3.1) by $(2, -1)$.



f. This is an example of cutting and pasting. Refer to 3.1, for graphing each piece of f:



3.5. A graph is **symmetric** about the x axis if its reflection through the x axis equals itself. Likewise for symmetry about the y axis or the origin.



For our famous graphs in 3.1, $y = |x|$ is symmetric about the y axis, and both $y = \frac{1}{x}$ and $y = x^3$ are symmetric about the origin. The graph $y^2 = x$ is symmetric about the x axis. A circle centered at the origin is symmetric in all three ways.

3.6. How to recognize symmetry from the equation of the graph.

(1) A graph is symmetric about the x axis if and only if replacing y with $-y$ in its equation gives an equivalent equation.

(2) A graph is symmetric about the y axis if and only if replacing x with $-x$ in its equation gives an equivalent equation.

(3) A graph is symmetric about the origin if and only if replacing y with $-y$ and x with $-x$ in its equation gives an equivalent equation.

For graphs of a function, we have a simpler characterization.

3.7. How to recognize symmetry of the graph of a function $y = f(x)$.

(1) The graph of a function $y = f(x)$ is never symmetric about the x axis unless $f(x) = 0$ for all x in the domain of f.

(2) The graph of a function $y = f(x)$ is symmetric about the y axis if and only if

$$f(-x) = f(x)$$

Included in this statement is $\{-x \mid x \text{ is in the domain of } f\}$ being equal to the domain of f.

(3) The graph of a function $y = f(x)$ is symmetric about the origin if and only if

$$f(-x) = -f(x)$$

Included in this statement is $\{-x \mid x \text{ is in the domain of } f\}$ being equal to the domain of f.

Examples 3.8. Discuss the symmetry of each of the following graphs; that is, state if it is symmetric about the x axis, the y axis, and/or the origin.

a. $y = (x^2 - 1)^3$.

b. $y = x^5 + 4x$.

c. $y = \frac{1}{x^3}$.

d. $y = x^5 + 4x + 1$.

e. $|y| = x$.

f. $|y| = |x|$.

g. $y^2 - x^2 = 1$.

Solutions. a. This is the graph of the function $f(x) = (x^2 - 1)^3$, so we can use 3.7.

$$f(-x) = ((-x)^2 - 1)^3 = (x^2 - 1)^3 = f(x),$$

so this graph is symmetric about the y axis, and not symmetric about the x axis or the origin.

b. This is the graph of the function $f(x) = x^5 + 4x$, so look at

$$f(-x) = (-x)^5 + 4(-x) = -(x^5 + 4x) = -f(x).$$

By 3.7, this graph is symmetric about the origin, and not symmetric about the x or y axis.

c. This is the graph of the function $f(x) = \frac{1}{x^3}$.

$$f(-x) = \frac{1}{(-x)^3} = -\frac{1}{x^3} = -f(x),$$

so by 3.7, this graph is symmetric about the origin and not symmetric about the x or y axis.

d. This is the graph of the function $f(x) = x^5 + 4x + 1$.

$$f(-x) = (-x)^5 + 4(-x) + 1 = -x^5 - 4x + 1.$$

Is this equal to $f(x)$? Then you'd have

$$x^5 + 4x + 1 = -x^5 - 4x + 1 \rightarrow x^5 + 4x = 0,$$

which is certainly not true for all x. Does $f(-x) = -f(x)$? Then you'd have

$$-x^5 - 4x + 1 = -(x^5 + 4x + 1) \rightarrow 1 = -1.$$

NO. So we have the graph of a function f that satisfies neither $f(-x) = f(x)$ nor $f(-x) = -f(x)$. By 3.7, this graph is symmetric about neither the origin nor the x or y axes.

e. I'll follow 3.6. Replacing x with $-x$ gives $|y| = -x$. This is not equivalent to $|y| = x$, since $-x \neq x$ for all x. So we do not have symmetry about the y axis. Replacing y with $-y$ gives $|-y| = x$. Since $|-y| = |y|$, this is equivalent to $|y| = x$, so we do have symmetry about the x axis. Replacing both y with $-y$ and x with $-x$ gives $|-y| = -x$ which is the same as $|y| = -x$—not equivalent.

By 3.6, our graph is symmetric about the x axis but not about the y axis or the origin.

f. Since $|-z| = |z|$ for any real z, following 3.6 exactly as in e. tells us that this graph is symmetric about the x axis, the y axis and the origin.

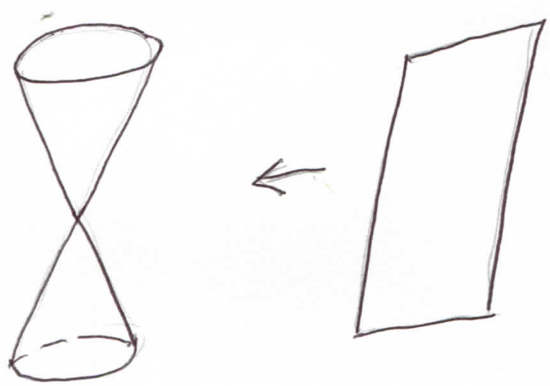
g. Since $(-z)^2 = z^2$ for any real z, 3.6 tells us that this graph has all the symmetries. ∎

3.9. The graph of any equation of the form

$$Ax^2 + Bx + C + Dy^2 + Ey = 0 \tag{3.10}$$

equating quadratic expressions in x and y is a **conic section**.

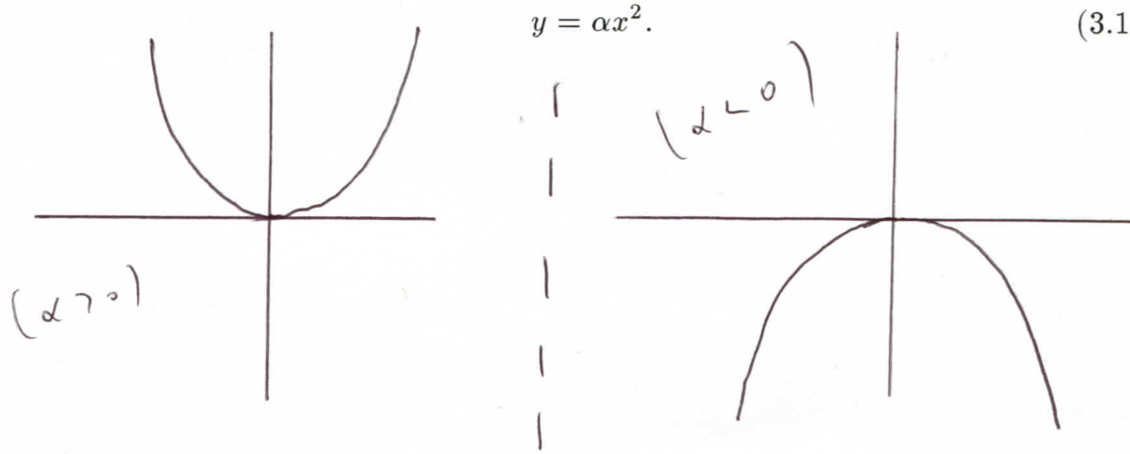
The name derives from the fact that these graphs are the intersection of a plane with a double cone.



After completing the square, in x and/or y, and making a translation (3.3a), (3.10) will have one of the following forms.

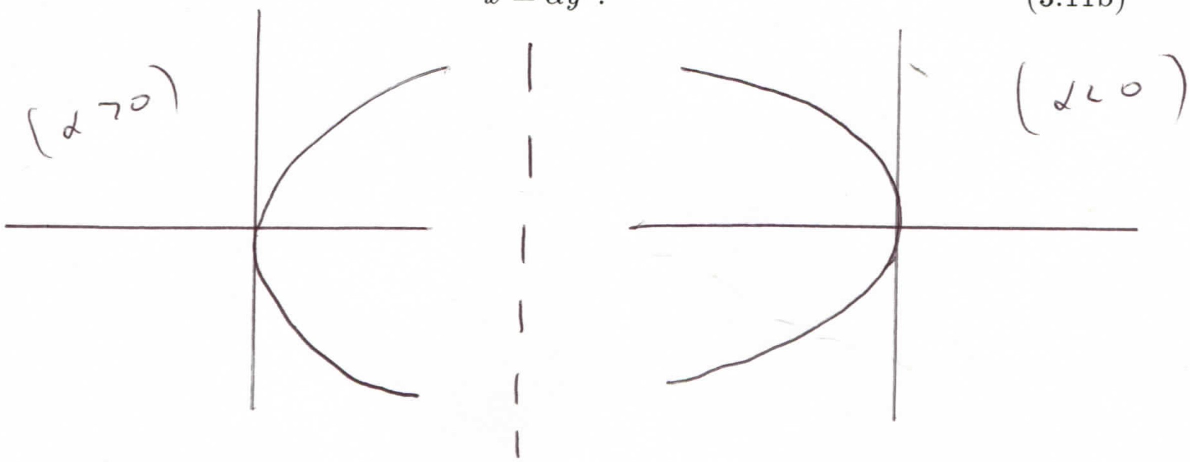
3.11. Parabola:

$$y = \alpha x^2. \tag{3.11a}$$



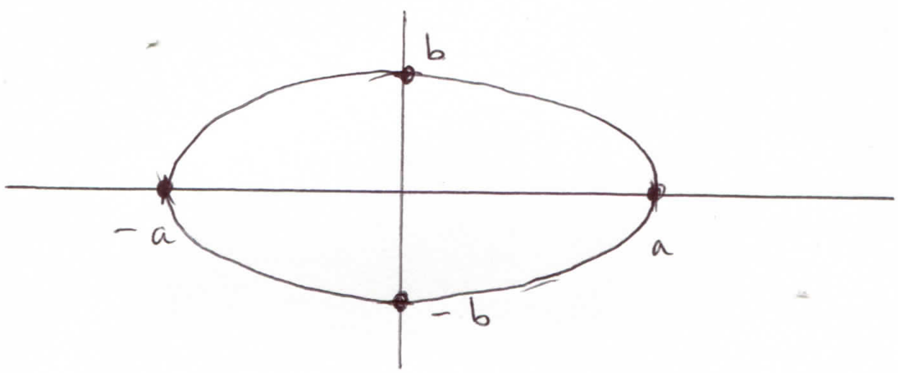
OR

$$x = \alpha y^2. \tag{3.11b}$$



3.12. Ellipse:

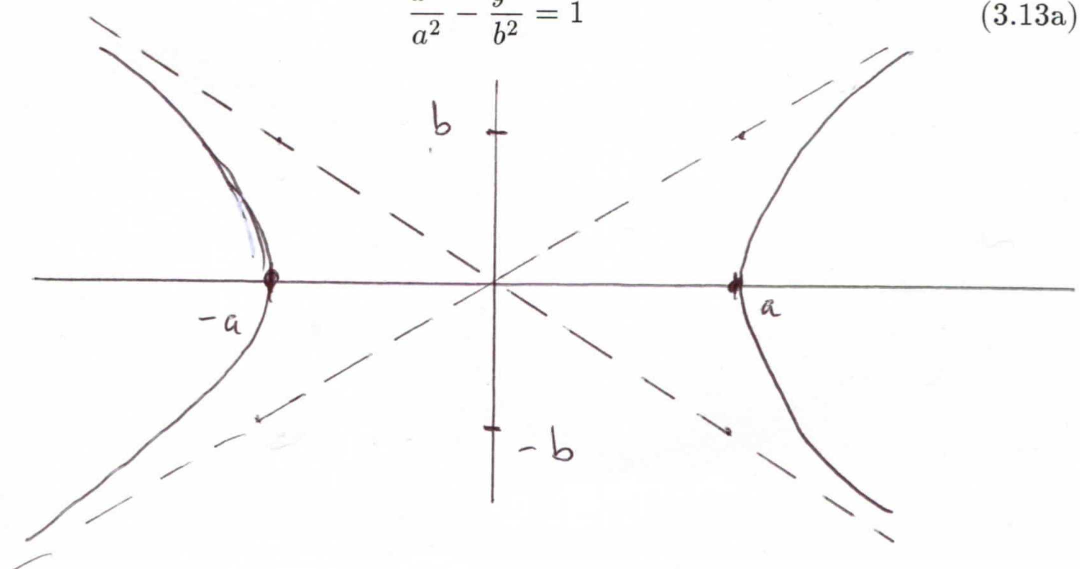
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



This is a stretched circle.

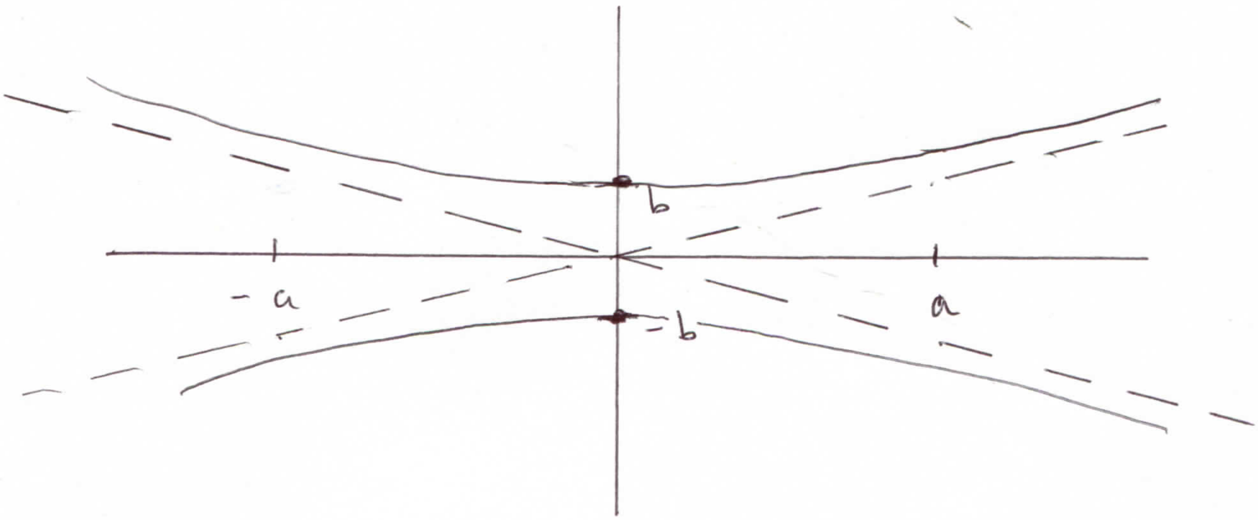
3.13. Hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{3.13a}$$



OR

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \tag{3.13b}$$



The dotted lines $y = \pm\frac{b}{a}x$ are **asymptotes**: lines that the graph approaches as $|x|$ or $|y|$ get large.

3.14. Examples. Graph each of the following.

1. $25x^2 + 150x + 16y^2 - 32y = 159.$
2. $x + 3y^2 - 12y + 17 = 0.$
3. $y^2 - 8y - 4x^2 = 20.$
4. $4x^2 + 8x - y^2 + 4y = 1.$

Solutions. 1. We must complete the square, both for x and y. Let's start with x:

$$x^2 + 6x = \frac{1}{25}\left(159 + 32y - 16y^2\right),$$

so

$$(x+3)^2 = x^2 + 6x + 9 = \frac{1}{25}\left(159 + 32y - 16y^2\right) + 9$$

or

$$25(x+3)^2 = 159 + 32y - 16y^2 + 225 = 384 + 32y - 16y^2$$

thus

$$y^2 - 2y = \frac{1}{16}\left(384 - 25(x+3)^2\right),$$

so

$$(y-1)^2 = y^2 - 2y + 1 = \frac{1}{16}\left(384 - 25(x+3)^2\right) + 1$$

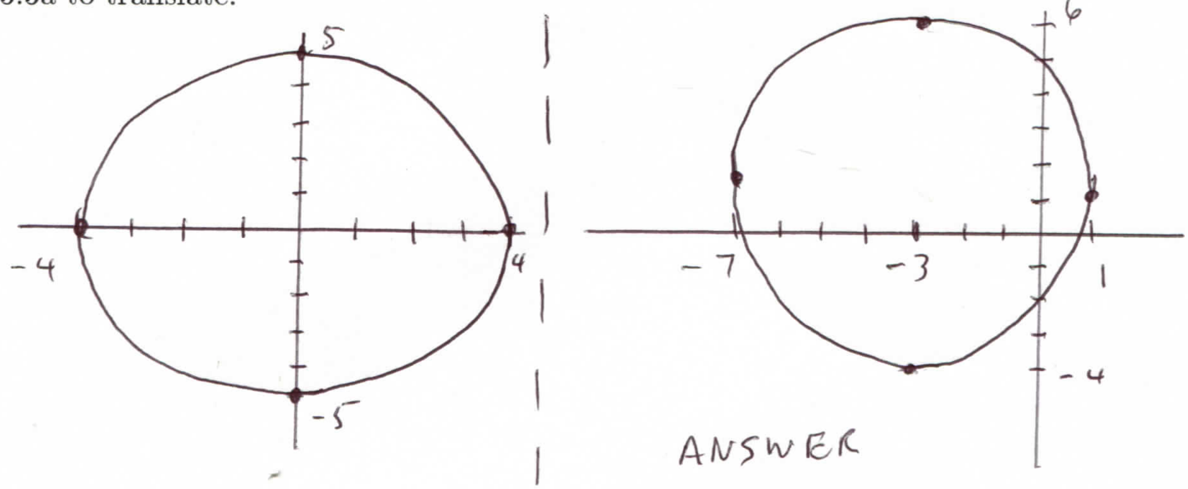
thus

$$16(y-1)^2 = 384 - 25(x+3)^2 + 16 = 400 - 25(x+3)^2.$$

This will look like 3.12 if we add $25(x+3)^2$ to both sides and divide both sides by 400:

$$\frac{(x+3)^2}{16} + \frac{(y-1)^2}{25} = 1.$$

To graph this, first graph $\frac{x^2}{16} + \frac{y^2}{25} = 1$ (this is 3.12, with $a = 4, b = 5$), then use 3.3a to translate.



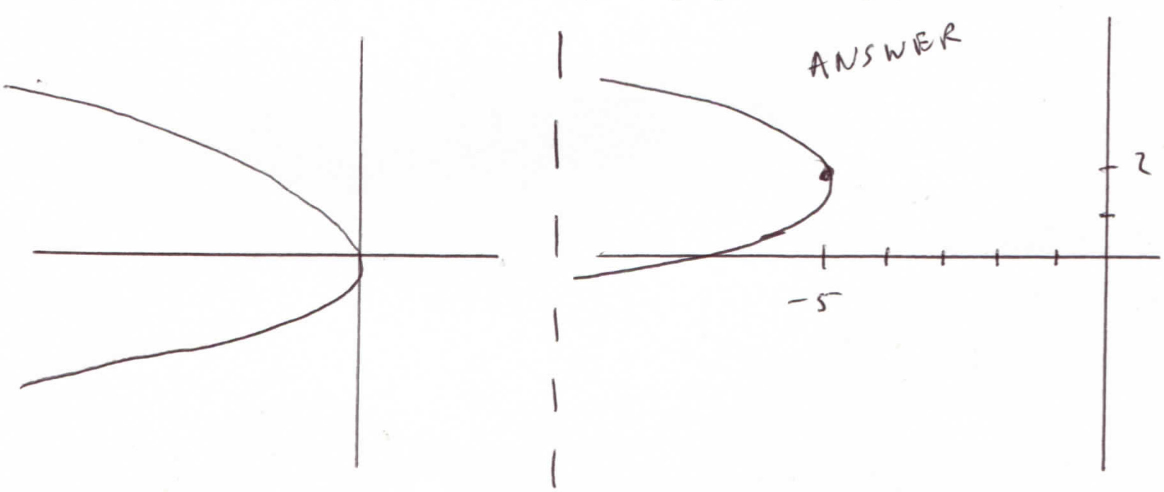
2. Here we only need to complete the square for y:

$$y^2 - 4y = \frac{1}{3}(-17 - x) \quad \to \quad (y-2)^2 = y^2 - 4y + 4 = \frac{1}{3}(-17 - x) + 4$$

so that

$$3(y-2)^2 = (-17 - x) + 12 = -5 - x \quad \to \quad (x+5) = -3(y-2)^2.$$

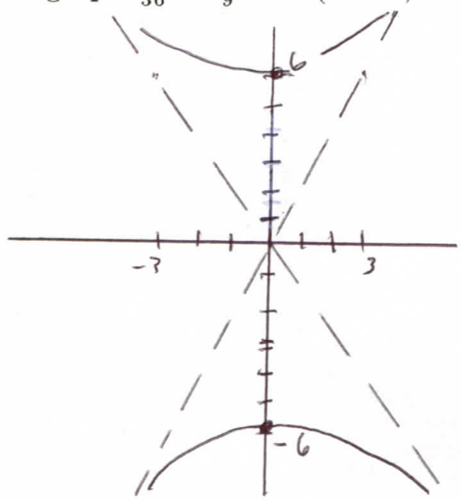
This looks like a sideways parabola 3.11b; first graph $x = -3y^2$, then translate.

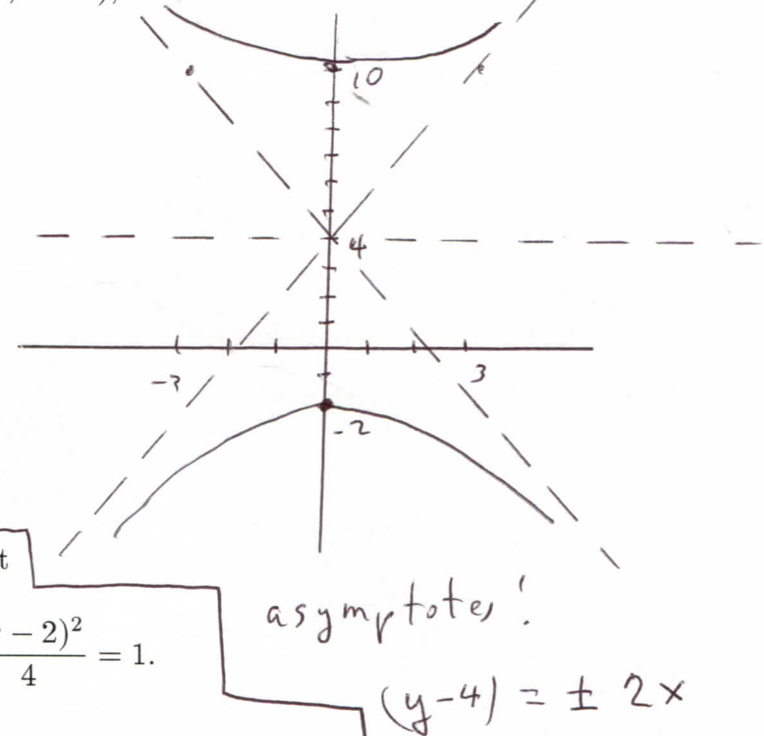


3. After completing the square for y as in 1, we get

$$\frac{(y-4)^2}{36} - \frac{x^2}{9} = 1.$$

First graph $\frac{y^2}{36} - \frac{x^2}{9} = 1$ (3.13b, with $b = 6, a = 3$), then translate.

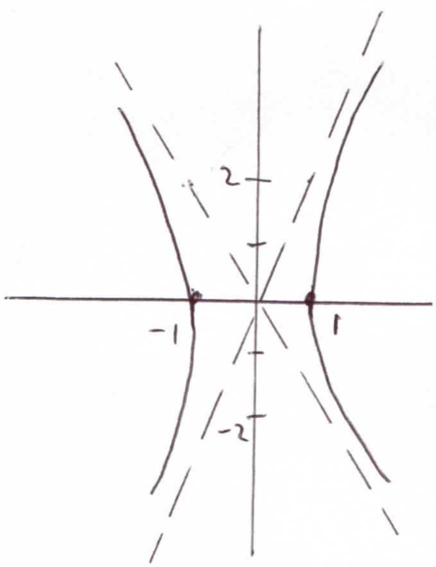


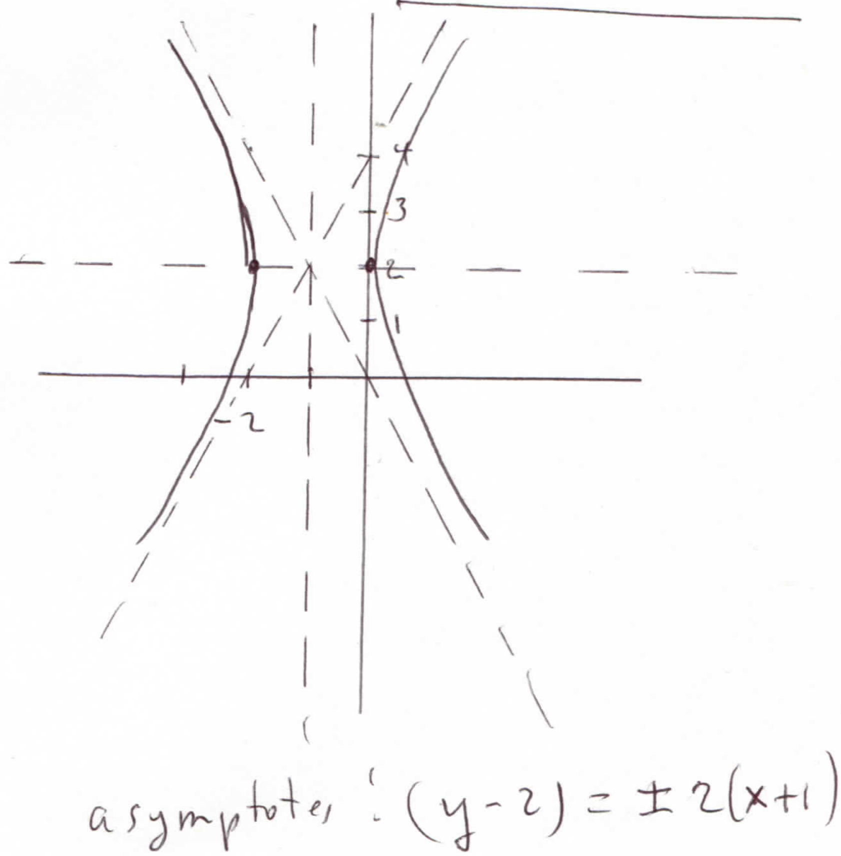


4. After completing squares as in 1, we get

$$(x+1)^2 - \frac{(y-2)^2}{4} = 1.$$

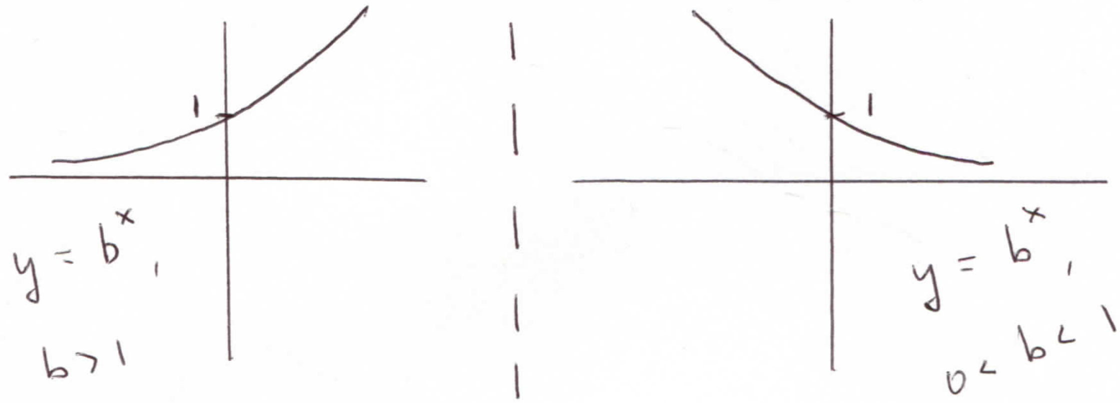
First graph $x^2 - \frac{y^2}{4} = 1$ (3.13a, with $a = 1, b = 2$), then translate.





IV. EXPONENTIAL AND LOGARITHMIC FUNCTIONS

4.1. The **exponential function,** with **base** b, is $f(x) = b^x$, where $b > 0, b \neq 1$. The domain of this function is the real line and the range is $(0, \infty)$.



In either case, $y = 0$ is a **horizontal asymptote** for $y = b^x$; this is a horizontal line that the graph approaches as $|x|$ gets large.

4.2. Population growth: If a population doubles every d years, then

$$P(t) = P(0)2^{\frac{t}{d}},$$

where $P(t)$ is the population t years from now. Note that $P(0)$ is the population now.

You should be able to verify that

$$P(t + d) = 2P(t)$$

for any time t; this is what it means to double every d years.

You should think about how you would change this formula if "doubling" were replaced by "tripling," "quadrupling," etc.

4.3. Radioactive decay: If a radioactive isotope has a half-life of h years, then

$$A(t) = A(0)2^{-\frac{t}{h}},$$

where $A(t)$ is the amount of the radioactive isotope t years from now. Note that $A(0)$ is the amount of isotope now.

You should be able to verify that

$$A(t + h) = \frac{1}{2}A(t)$$

for any time t; this is what is meant by half-life.

4.4. Examples.

a. Suppose a population doubles every 9 years. If there are 27 now, how many will there be in 92 years?

b. Suppose a radioactive isotope has a half-life of 7 minutes. If there are 3 pounds of it now, how much will there be in an hour?

Solutions.

a. If $P(t)$ is the population after t years, then by 4.2, with $d = 9, P(0) = 27$,

$$P(t) = 27(2)^{\frac{t}{9}},$$

so $P(92) = 27(2)^{\frac{92}{9}}$.

b. If $A(t)$ is the amount of the isotope t minutes from now, then by 4.3, with $h = 7, A(0) = 3$,

$$A(t) = 3(2)^{-\frac{t}{7}},$$

so $A(60) = 3(2)^{-\frac{60}{7}}$.

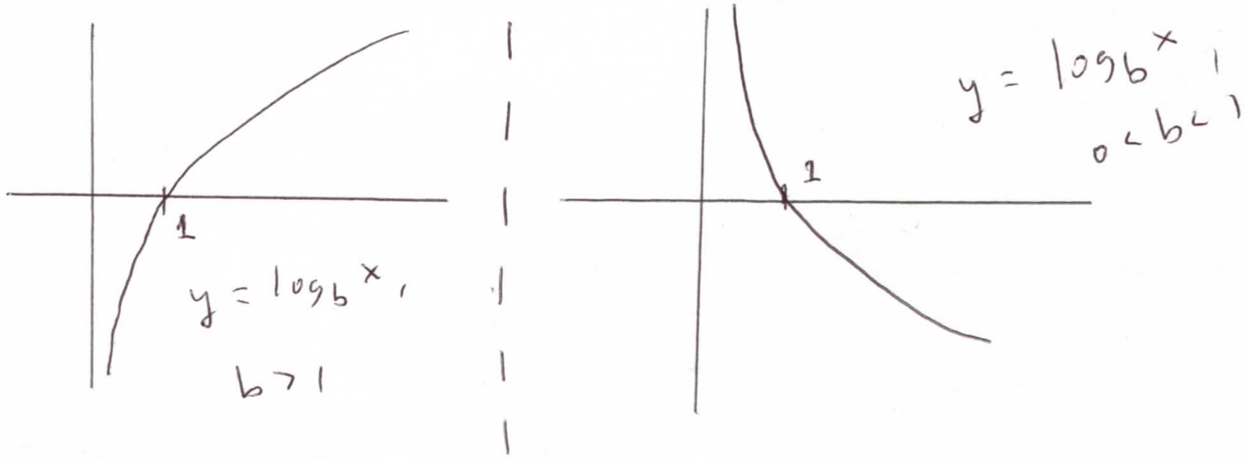
4.5. Logarithm with base b, $\log_b$, is the inverse function (see 2.38) of the exponential function $f(x) \equiv b^x$; that is,

$$b^{\log_b x} = x, \tag{4.6}$$

for x positive, and

$$\log_b(b^x) = x, \tag{4.7}$$

for any real x. The domain of $\log_b$ is $(0, \infty)$ and the range is the real line.



For this graph, $x = 0$ is a **vertical asymptote**; this is a vertical line that the graph approaches as $|y|$ gets large.

Examples 4.8. Find the following logarithms: $\log_2(8), \log_{10}(100), \log_4(2), \log_{25}(\frac{1}{5})$ and $\log_{1,000}(.01)$.

Solutions. By (4.7), $\log_2(8) = \log_2(2^3) = 3; \log_{10}(100) = \log_{10}(10^2) = 2$. Note that I am asking "What power of 2 is 8? What power of 10 is 100?"

For $\log_4(2)$, I am asking "What power of 4 is 2?" Perhaps you know this immediately, but I am going to be cautious. Give this number we're looking for a name:

$$y = \log_4(2).$$

We can get rid of the mysterious log with (4.6), which tells us that

$$4^y = 4^{\log_4(2)} = 2.$$

Next, I'd like both sides of the equation to be powers of 2, so that I can equate exponents. Write 4 as a power of 2, in the previous equation, so that

$$2^1 = 2 = 4^y = (2^2)^y = 2^{2y},$$

so that the exponents 1 and $2y$ must be equal: $1 = 2y$, so that our answer is $y = \frac{1}{2}$; $\log_4(2) = \frac{1}{2}$.

You can always check your logarithm. Here we are asserting that $4^{\frac{1}{2}} = 2$, which it does; our answer checks.

For $\log_{25}(\frac{1}{5})$, again give it a name: $y \equiv \log_{25}(\frac{1}{5})$. Again use (4.6):

$$25^y = 25^{\log_{25}(\frac{1}{5})} = \frac{1}{5};$$

we can express both 25 and $\frac{1}{5}$ as powers of 5:

$$(5^2)^y = 5^{-1}, \quad \text{so} \quad 5^{2y} = 5^{-1},$$

so that $2y = -1$ and $y = -\frac{1}{2}$. Our answer is $\log_{25}(\frac{1}{5}) = -\frac{1}{2}$.
CHECK: does $25^{-\frac{1}{2}} = \frac{1}{5}$? You should be able to verify this.

Again, for our last logarithm, write $y = \log_{1,000}(.01)$; then, by (4.6),

$$1,000^y = 1,000^{\log_{1,000}(.01)} = .01, \quad \text{or} \quad 10^{3y} = (10^3)^y = 1,000^y = .01 = 10^{-2},$$

so that $3y = -2, y = -\frac{2}{3}$.

We asked ourselves "What power of 1,000 is .01?" and concluded that our answer is $\log_{1,000}(.01) = -\frac{2}{3}$, because $1,000^{-\frac{2}{3}} = .01$.

Examples 4.9. Solve each of the following for x:
a. $2^x = 5$. b. $\log_{10} x = 3$. c. $\log_8(\frac{x}{3}) = -1$. d. $\log_x(64) = 3$.
e. $1,000,000 = 100 \log_2 x$. f. $80 = 5(3^{4x}) - 20$.

Solutions. a. We need to get x out of the exponent. This is done with a log, that is, by taking $\log_2$ of both sides of the equation: by (4.7),

$$x = \log_2(2^x) = \log_2(5).$$

b. Get rid of the log by exponentiating, that is, raising 10 to each side of the equation: (4.6) tells us that

$$x = 10^{\log_{10}(x)} = 10^3 = 1,000.$$

c. As in b, get rid of the log by exponentiating; since we have a $\log_8$, raise 8 to each side: by (4.6),

$$\frac{x}{3} = 8^{\log_8(\frac{x}{3})} = 8^{-1} = \frac{1}{8},$$

so $x = \frac{3}{8}$.

d. By (4.6),

$$64 = x^{\log_x(64)} = x^3,$$

so $x = (64)^{\frac{1}{3}} = 4$.

e. *Before* exponentiating, isolate the log, by dividing both sides of the equation by 100, so that

$$10,000 = \log_2(x).$$

By (4.6),

$$x = 2^{\log_2(x)} = 2^{10,000}.$$

f. We must first isolate 3^{4x}. Add 20 to both sides, to get $100 = 5(3^{4x})$, then divide both sides by 5, so that

$$20 = 3^{4x}.$$

By (4.7),

$$4x = \log_3(3^{4x}) = \log_3(20),$$

$x = \frac{1}{4}\log_3(20)$.

4.10. Properties of logarithms:

(1) $\log_b(xy) = \log_b x + \log_b y$

(2) $\log_b(x^r) = r\log_b x$

Notice that these are the opposites of the laws for exponents (1.43)

Examples 4.11. a. Simplify $\log_2(\frac{1}{1,000})$.

b. Simplify $\frac{1}{3}\log_{10}(64) - \frac{1}{2}\log_{10}(\frac{1}{9})$.

c. Solve for x: $\log_8(x) + \log_8(x-2) = 1$.

d. Solve for x: $2\log_3(x) - \log_3(x+1) = 0$.

Solutions. a. This is $\log_2(10^{-3}) = -3\log_2(10)$, by 4.10(2).

b. This is $\frac{1}{3}\log_{10}(2^6) - \frac{1}{2}\log_{10}(3^{-2}) = \log_{10}(2^2) + \log_{10}(3)$, by 4.10(2), which equals $\log_{10}(2^2 \cdot 3) = \log_{10}(12)$, by 4.10(1).

c. First get the left-hand side entirely inside a single log. By 4.10(1),

$$1 = \log_8(x) + \log_8(x-2) = \log_8(x(x-2)),$$

so now use (4.6) to get rid of the $\log_8$:

$$x(x-2) = 8^{\log_8(x(x-2))} = 8^1 = 8,$$

thus we have a quadratic to solve: $x^2 - 2x - 8 = 0, (x-4)(x+2) = 0, x = 4, -2$.
Only $x = 4$ is defined, since the domain of $\log_8$ is $(0, \infty)$.

d. As in c, start by putting all logs together, using 4.10((2) followed by (1)):

$$0 = 2\log_3(x) - \log_3(x+1) = \log_3(x^2) + \log_3((x+1)^{-1}) = \log_3(\frac{x^2}{x+1}).$$

Next get rid of the $\log_3$ with (4.6):

$$\frac{x^2}{x+1} = 3^{\log_3(\frac{x^2}{x+1})} = 3^0 = 1;$$

thus $x^2 = x + 1$, so that $x^2 - x - 1 = 0$. Now use the quadratic formula ((1.33)) to get $x = \frac{1}{2}(1 \pm \sqrt{5})$.
Only $x = \frac{1}{2}(1 + \sqrt{5})$ is defined, since $\frac{1}{2}(1 - \sqrt{5})$ is negative.

Examples 4.12. In all of these, simplify your answer.

a. Suppose a population triples every 7 years and is currently 1,000. When will it reach 10,000?

b. Suppose a substance has a half-life of 3 hours. If there is 4 grams of it now, how long will it take to shrink to .01 grams?

c. Suppose a fossil contains 3% of the C-14 it had when it died. How old is the fossil? (C-14 has half-life of 5570 years.)

Solutions. a. If $P(t)$ is the population t years from now, then by 4.2, with $d = 7$, $P(0) = 1,000$, and doubling replaced by tripling,

$$P(t) = 1,000(3^{\frac{t}{7}}).$$

We are being asked "for what t will $P(t)$ equal 1,000?" So set $P(t) = 10,000$, and solve for t. First isolate the exponential by dividing both sides by $1,000 : 10 = 3^{\frac{t}{7}}$, then use (4.7) to get rid of the exponent:

$$\frac{t}{7} = \log_3(3^{\frac{t}{7}}) = \log_3(10),$$

so that our desired time is $t = 7\log_3(10)$ years from now.

b. Let $A(t)$ be the number of grams of this substance t hours from now. By 4.3, with $A(0) = 4$ and $h = 3$,

$$A(t) = 4(2^{-\frac{t}{3}}).$$

We want t so that $A(t) = .01$. Thus we set $.01 = A(t)$, so that, dividing both sides by 4, we have $\frac{1}{400} = 2^{-\frac{t}{3}}$, and by (4.7),

$$-\frac{t}{3} = \log_2(2^{-\frac{t}{3}}) = \log_2(\frac{1}{400}),$$

thus $t = -3\log_2(\frac{1}{400})$. This simplifies, by 4.10(2):

$$-3\log_2(\frac{1}{400}) = -3\log_2(400^{-1}) = 3\log_2(400) = 3\log_2(20^2) = 6\log_2(20).$$

Our answer is $6\log_2(20)$ hours from now.

c. Let $A(t)$ be the amount of C-14 t years after death. Let T be the age of the fossil, in years (that means years since death). We are told that $A(T)$, the amount of C-14 now, is 3% of $A(0)$, the amount of C-14 at the moment of death; that is,

$$A(T) = .03A(0).$$

Well, by 4.3, with $h = 5570$,

$$A(T) = [A(0)]\,(2^{-\frac{T}{5570}}),$$

thus $.03 = (2^{-\frac{T}{5570}})$, and we may solve for T with the help of (4.7):

$$-\frac{T}{5570} = \log_2(2^{-\frac{T}{5570}}) = \log_2(.03),$$

so that $T = -5570\log_2(.03)$ years. This can be simplified using 4.10(2):

$$T = -5570\log_2(\frac{3}{100}) = -5570\log_2((\frac{100}{3})^{-1}) = 5570\log_2(\frac{100}{3}).$$

Our answer is $5570\log_2(\frac{100}{3})$ years old.

4.13. Famous irrational number e: A feeble attempt at motivation and a definition. For small h, consider the average rate of change of $f(x) \equiv b^x$ over $[a, a+h]$:

$$\frac{f(a+h) - f(a)}{h} = \frac{b^{a+h} - b^a}{h} = b^a\left(\frac{b^h - 1}{h}\right) = f(a)\left(\frac{b^h - 1}{h}\right).$$

As h shrinks to 0, that fraction

$$\left(\frac{b^h - 1}{h}\right)$$

gets closer and closer to a number (it turns out to be $\ln b$; see 4.15). There is a certain famous irrational number, e, such that that number is 1; that is,

$$\left(\frac{e^h - 1}{h}\right)$$

gets closer and closer to 1 as h shrinks to 0.

In the language of calculus, this is saying that $f(x) \equiv e^x$ is a function whose instantaneous rate of change (derivative) equals itself.

It can also be shown (with calculus) that $(1 + \frac{1}{n})^n$ gets closer and closer to e as n gets larger and larger. This interpretation is relevant in banking; compound interest compounded more and more frequently becomes "compounded instantaneously."

4.14. All you need to know about e: It is a famous irrational number between 2 and 3. You will hear a lot about it in calculus, including a real definition.

4.15. Most desired logarithm and exponential: The **natural logarithm** is $\ln \equiv \log_e$. **The exponential function** is the exponential function with base e; that is, $f(x) \equiv e^x$.

Can Show: For any base b, $\log_b x = \frac{\ln x}{\ln b}$, for any positive x.

Note that (4.6) and (4.7), with $b \equiv e$, become

$$e^{\ln x} = x, \tag{4.16}$$

for x positive, and

$$\ln(e^x) = x, \tag{4.17}$$

for any real x.

4.18. Examples.

a. Simplify each of the following: (i) $\frac{\ln(\frac{1}{2})}{\ln(\frac{1}{4})}$; (ii) $\frac{\ln 64}{\ln 16}$; (iii) $\log_2(16)$; (iv) $\log_3(9\sqrt{3})$; (v) $e^{2\ln x}$; (vi) $\log_{\frac{1}{9}} 3$; (vii) $\log_2(\frac{1}{4})$; (viii) $\ln(\sqrt{e})$

b. In each of the following, solve for the variable: (i) $5(2^x) = 20$; (ii) $2(3^{9t}) = 18$; (iii) $50(\frac{1}{2})^{\frac{y}{10}} = 10$; (iv) $10 + 9e^{-2t\ln 5} = 13$; (v) $5e^{2t} + 7 = 22$; (vi) $\ln(2x^2) - \ln x = 1$.

Solutions. a. (i) $\frac{\ln(\frac{1}{2})}{\ln(\frac{1}{4})} = \frac{\ln(2^{-1})}{\ln(2^{-2})} = \frac{-\ln 2}{-2\ln 2} = \frac{1}{2}$, by 4.10(2).

(ii) $\frac{\ln 64}{\ln 16} = \frac{\ln(2^6)}{\ln(2^4)} = \frac{6\ln 2}{4\ln 2} = \frac{6}{4} = \frac{3}{2}$, by 4.10(2).

(iii) $\log_2(16) = \log_2(2^4) = 4$, by (4.7).

(iv) $\log_3(9\sqrt{3}) = \log_3(3^{2+\frac{1}{2}}) = \frac{5}{2}$, by (4.7).

(v) We can't cancel e and ln, as in (4.16), yet, because of that 2 being in the way. So first rewrite $2\ln x$ as $\ln(x^2)$, by 4.10(2):

$$e^{2\ln x} = e^{\ln(x^2)} = x^2,$$

by 4.10(2) followed by (4.16).

(vi) Let $\log_{\frac{1}{9}} 3 \equiv y$, then we get rid of logs with (4.6):

$$(\frac{1}{9})^y = (\frac{1}{9})^{\log_{\frac{1}{9}} 3} = 3,$$

so $3^1 = (3^{-2})^y = 3^{-2y}$, so that $1 = -2y$ and our desired y is $-\frac{1}{2}$.

(vii) $\log_2(\frac{1}{4}) = \log_2(2^{-2}) = -2$, by (4.7).

(viii) $\ln(\sqrt{e}) = \ln(e^{\frac{1}{2}}) = \frac{1}{2}$, by (4.7).

b. (i) First isolate the 2^x, by dividing both sides by 5: $2^x = 4 = 2^2$, so $x = 2$.

(ii) $3^{9t} = 9 = 3^2$, so $9t = 2, t = \frac{2}{9}$.

(iii) $(\frac{1}{2})^{\frac{y}{10}} = \frac{1}{5}$, so apply $\log_2$ to both sides; by 4.10(2),

$$\frac{y}{10}\log_2(\frac{1}{2}) = \log_2\left(\left(\frac{1}{2}\right)^{\frac{y}{10}}\right) = \log_2(\frac{1}{5}),$$

thus

$$y = 10\frac{\log_2(\frac{1}{5})}{\log_2(\frac{1}{2})} = 10\frac{-\log_2(5)}{-\log_2(2)} = 10\frac{-\log_2(5)}{-1} = 10\log_2 5.$$

(iv) $9e^{-2t\ln 5} = 3$, so $e^{-2t\ln 5} = \frac{1}{3}$, and we get rid of the e with (4.17):

$$-2t\ln 5 = \ln(e^{-2t\ln 5}) = \ln(\frac{1}{3}) = -\ln 3,$$

thus $t = \frac{\ln 3}{2\ln 5}$.

(v) $5e^{2t} + 7 = 22$; then $e^{2t} = 3$, so $(2t) = \ln(e^{2t}) = \ln 3, t = \frac{1}{2}\ln 3$.

(vi) By 4.10,

$$1 = \ln(2x^2) - \ln x = \ln(2x^2) + \ln(x^{-1})) = \ln((2x^2)(x^{-1}) = \ln(2x),$$

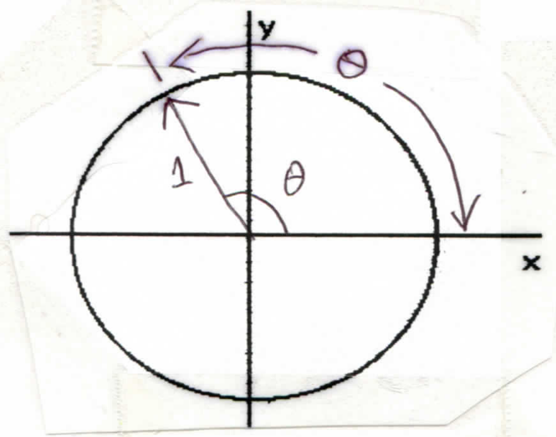
thus by (4.16),

$$e = e^1 = e^{\ln(2x)} = 2x,$$

so that $x = \frac{e}{2}$.

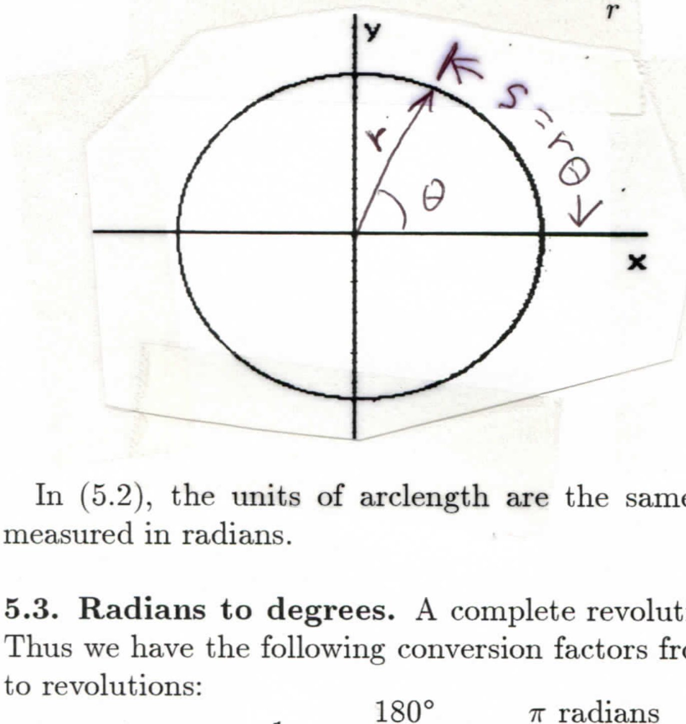
V. TRIGONOMETRY.

5.1. Radian measure of angles. The **unit circle** is a circle of radius one, centered at the origin $(0, 0)$. θ **radians**, measured from the positive x-axis, is the angle that produces an arclength of θ on the unit circle, starting at $(1, 0)$. Counter-clockwise is positive.



If we shrink or expand the unit circle, we maintain the ratio between the radius r and the arclength s:

$$\theta = \frac{s}{r} \tag{5.2}$$



In (5.2), the units of arclength are the same as the units of radius and θ is measured in radians.

5.3. Radians to degrees. A complete revolution is 360° and is also 2π radians. Thus we have the following conversion factors from radians to degrees and radians to revolutions:

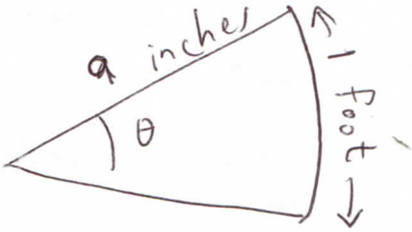
$$1 = \frac{180°}{\pi \text{ radians}} = \frac{\pi \text{ radians}}{180°} = \frac{2\pi \text{ radians}}{\text{revolution}}.$$

Examples 5.4. a. How many degrees is $\frac{5\pi}{4}$ radians?

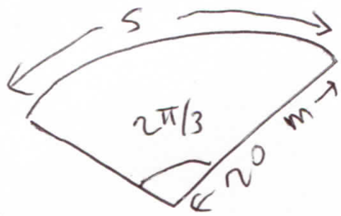
b. How many radians is 160 degrees?

c. How many radians is a right angle?

d. Find θ in the following picture.



e. Find the length of the arc in the following picture.



Solutions. a. By 5.3,

$$\frac{5\pi}{4} \text{ radians } = \left[\frac{5\pi}{4} \text{ radians}\right]\left[\frac{180°}{\pi \text{ radians}}\right] = 225°.$$

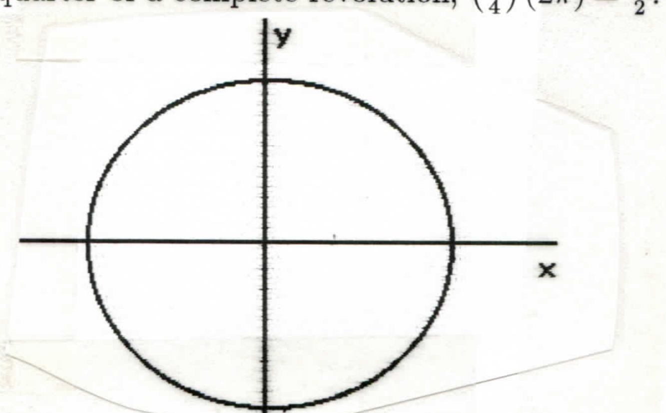
b. By 5.3, with the conversion factor inverted,

$$160° = [160°]\left[\frac{\pi \text{ radians}}{180°}\right] = \frac{8\pi}{9} \text{ radians} .$$

c. A right angle is 90°, so we want

$$90° = [90°]\left[\frac{\pi \text{ radians}}{180°}\right] = \frac{\pi}{2} \text{ radians} .$$

This is one-quarter of a complete revolution; $\left(\frac{1}{4}\right)(2\pi) = \frac{\pi}{2}$.



d. Here s equals one foot, and r equals nine inches (from (5.2)). We must have the same units for s and r, so I will write s as twelve inches, and use (5.2):

$$\theta = \frac{12}{9} \text{ radians } = \frac{4}{3} \text{ radians} .$$

e. Here r equals 20 meters and θ equals $\frac{2\pi}{3}$ radians, so by (5.2) the arclength s is given by

$$s = \theta r = \left[\frac{2\pi}{3} \text{ radians}\right] [20 \text{ meters }] = \frac{40\pi}{3} \text{ meters} .$$

Note how the radians disappeared after being multiplied by distance: (radians)(meters) equals meters. ∎

5.5. Angular speed measures the rate of rotation of a disc. If a disc rotates about its axis at a constant rate, turning θ radians in a time t, then the angular speed, ω, of the rotation, is

$$\omega \equiv \frac{\theta}{t}.$$

Let's use the term **linear speed** for the more familiar (linear) distance divided by time. For a point on the circumference of a rotating disc, its linear speed v is the arclength it has moved divided by time:

$$v = \frac{s}{t}.$$

Dividing both sides of (5.2) by time gives us

$$\omega = \frac{v}{r}, \tag{5.6}$$

where v is the linear speed of a point on the circumference of the disc, r is the radius of the disc and ω is the angular speed of rotation, in radians per unit time.

In (5.6) as in (5.2), the units of arclength are the same as the units of radius.

Examples 5.7. a. Suppose a disc of radius 5 feet is rotating at 60 revolutions per minute. At what (linear) speed is a bug clinging to the circumference of the disc moving?

b. Suppose you are travelling in your car at 60 miles per hour. If each of your tires has a radius of two feet, at what (angular) speed are your tires revolving?

c. In b., how many times will each of your tires make a complete revolution in five minutes?

d. If you stand still on the equator, at what speed are you moving in space? (Assume the diameter of the earth is 8,000 miles.) Disregard the motion of the earth around the sun and any motion of our solar system.

Solutions. a. First we must change the angular speed to radians per minute. Since there are 2π radians in a revolution, the angular speed is

$$\omega = 120\pi \text{ radians per minute.}$$

By (5.6), the linear speed on the circumference of the disc is

$$v = \omega r = [120\pi \text{ radians per minute}] [5 \text{ feet }] = 600\pi \text{ feet per minute.}$$

Note how, as in Example 5.4e, the radians disappear after they are multiplied by distance.

b. Here $v = 60$ miles per hour and r equals two feet. We must make our units the same. Let's write v as one mile per minute. Since there are 5,280 feet in a mile, we can also express v as 5,280 feet per minute, so that, by (5.6),

$$\omega = \frac{5,280 \text{ feet per minute}}{2 \text{ feet}} = 2,640 \text{ radians per minute} .$$

Notice how radians appear as $\frac{\text{feet}}{\text{feet}}$.

c. We just found that, in one minute, each of your tires revolves through 2,640 radians. Since there are 2π radians in a revolution, that's

$$\frac{2,640}{2\pi} = \frac{1,320}{\pi} \text{ revolutions per minute} .$$

In five minutes, you'll have

$$5\left(\frac{1,320}{\pi}\right) = 6,600 \text{ revolutions} .$$

d. The motion I'm asking you to worry about is the revolution of the earth on its axis. The earth makes a complete revolution every 24 hours. That's 2π radians every 24 hours, or

$$\omega = \frac{2\pi \text{ radians}}{24 \text{ hours}} = \frac{\pi}{12} \text{ radians per hour} .$$

Then I'm giving you an approximation of r as 4,000 miles. By (5.6),

$$v = r\omega = (4,000 \text{ miles })(\frac{\pi}{12} \text{ radians per hour }) = \frac{1,000\pi}{3} \text{ miles per hour} .$$

That's more than 1,000 miles per hour. It sounds breathtaking. ∎

5.8. Area of sector of angle θ radians, radius r, is

$$\frac{\theta}{2}r^2.$$

This follows from the fact that the area of the entire disc of radius r, which is the sector of angle 2π radians, has area πr^2; to get the area of the sector of angle θ, multiply πr^2 by $\frac{\theta}{2\pi}$.

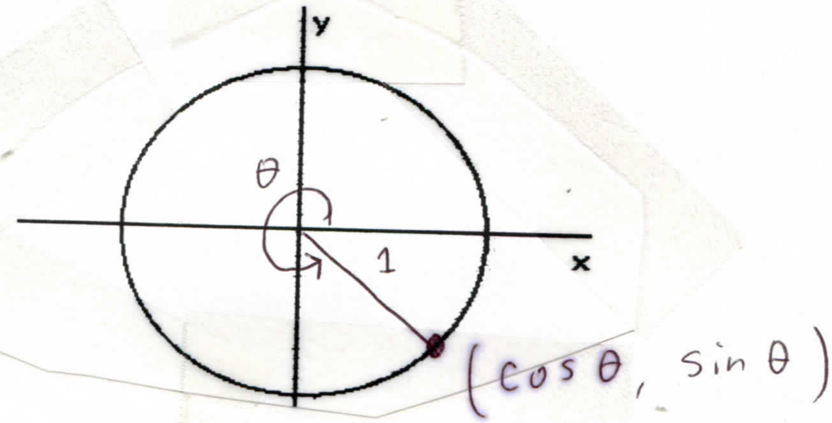
Example 5.9. Find the area of a pizza slice of angle $\frac{\pi}{6}$ radians, if the radius of the pizza is two feet.

Solution. In 5.8, r is 2 feet and θ is $\frac{\pi}{6}$ radians, so the desired area is

$$\left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right)(2 \text{ feet })^2 = \left(\frac{\pi}{3}\right) \text{ feet }^2.$$

∎

5.10. The sine and cosine functions: Measure θ radians on the unit circle, counterclockwise from the positive x-axis (see 5.1). The point on the unit circle at the end of this arc has coordinates $(\cos\theta, \sin\theta)$.



Here is a relationship between sin and cos that follows immediately from the picture defining them.

5.11 Pythagorean theorem for sine and cosine:

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{for any real } \theta.$$

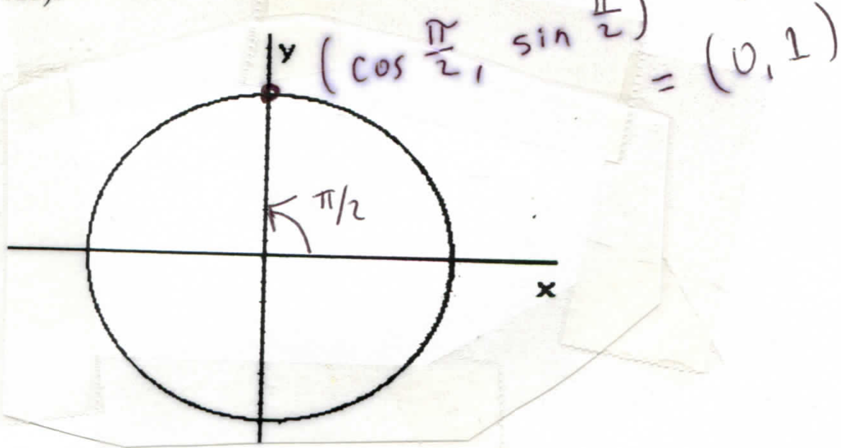
The custom is to write $\sin^2\theta$ for $(\sin\theta)^2$, likewise for cosine.

Examples 5.12. a. Find $\sin 0, \sin(\frac{\pi}{2}), \sin(\pi), \sin(\frac{3\pi}{2}), \sin(2\pi)$, and $\sin(\frac{5\pi}{2})$.

b. Find $\cos(17\pi)$, $\sin(-\frac{9\pi}{2})$ and $\cos(2,000,000\pi)$.

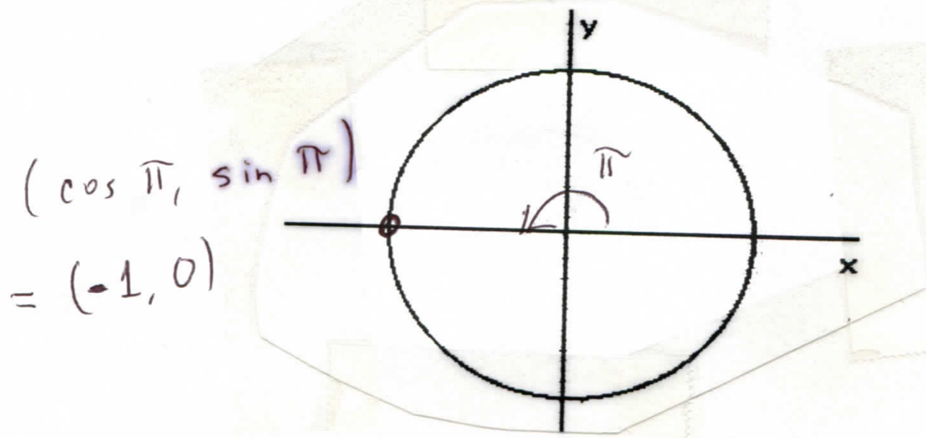
c. Suppose $\sin(\theta) = \frac{1}{3}$ and $\frac{5\pi}{2} \leq \theta \leq \frac{7\pi}{2}$. What is $\cos(\theta)$?

Solutions. a. The important thing is to locate the desired angle on the unit circle, as in 5.10. For example, to evaluate $\sin(\frac{\pi}{2})$, I notice that 2π makes a complete revolution of the circle, so $\frac{\pi}{2}$ is one quarter of the way around the unit circle (see also Example 5.4c).



The point I just marked on the unit circle is $(0, 1)$. That tells us that $0 = \cos(\frac{\pi}{2})$ and $1 = \sin(\frac{\pi}{2})$.

Similarly, π radians is half-way around the unit circle.



The point on the unit circle corresponding to π radians is $(-1, 0)$. This means that $-1 = \cos(\pi)$ and $0 = \sin(\pi)$. We always get both sine and cosine simultaneously this way.

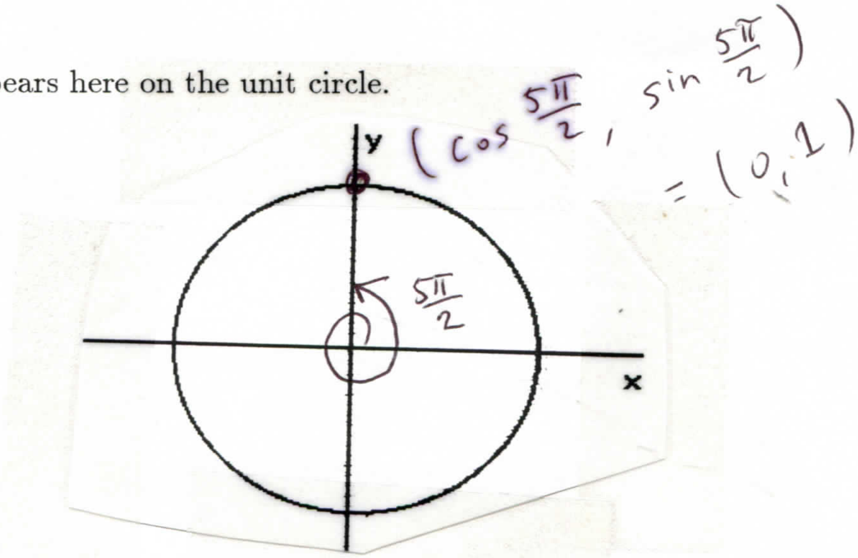
I will leave it to you, to draw or believe, that the point on the unit circle corresponding to 0 radians is $(1, 0)$, so that $\cos(0) = 1, \sin(0) = 0$; the point on the unit circle corresponding to $\frac{3\pi}{2}$ radians is $(0, -1)$, so that $\cos(\frac{3\pi}{2}) = 0, \sin(\frac{3\pi}{2}) = -1$.

The angle 2π radians brings you all the way around the unit circle back to $(1, 0)$ again. Thus $\cos(2\pi) = 1, \sin(2\pi) = 0$. In fact, for *any* integer k, $\cos(2k\pi) = 1, \sin(2k\pi) = 0$; you make k complete revolutions of the unit circle and end up back where you started, at $(1, 0)$.

For the angle $\frac{5\pi}{2}$, write that as

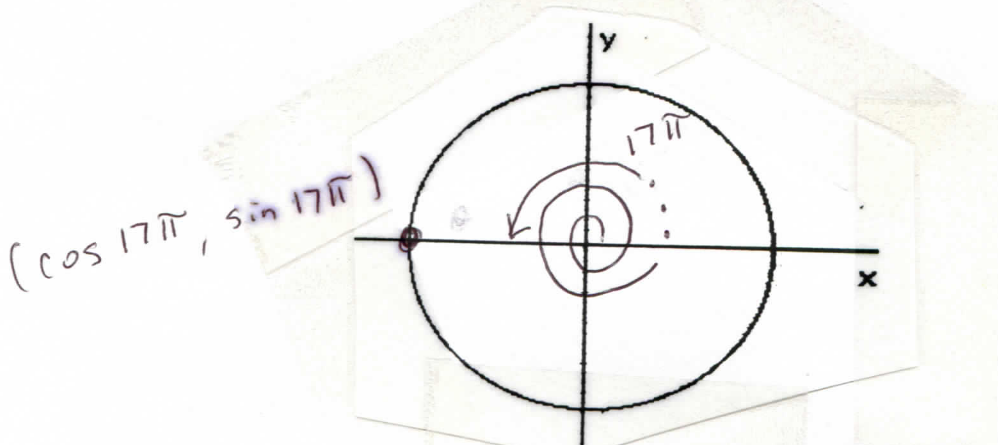
$$\frac{5\pi}{2} = \frac{(4+1)\pi}{2} = 2\pi + \frac{\pi}{2},$$

thus $\frac{5\pi}{2}$ appears here on the unit circle.



This means that $(\cos(\frac{5\pi}{2}), \sin(\frac{5\pi}{2})) = (\cos(\frac{\pi}{2}), \sin(\frac{\pi}{2})) = (0, 1)$; in particular, $\sin(\frac{5\pi}{2}) = 1$.

b. To locate the angle of 17π radians on the unit circle, we need to subtract integral multiples of 2π; that corresponds to making some complete revolutions around the unit circle. 16π is the nearest integral multiple of 2π. Since $17\pi = 16\pi + \pi = 8(2\pi) + \pi$, here's where the angle of 17π radians appears on the unit circle.

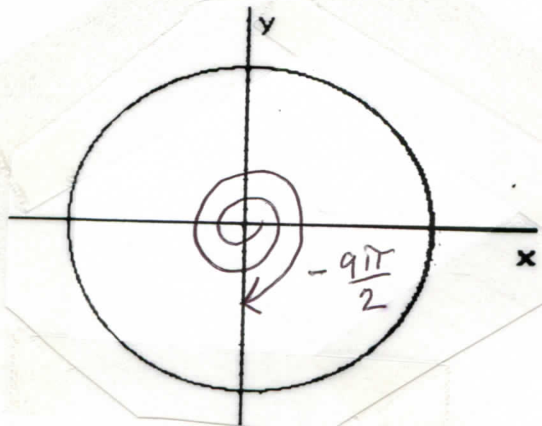


The point on the unit circle corresponding to that angle is $(-1, 0)$. Thus $\cos(17\pi) = -1$.

To locate the angle of $-\frac{9\pi}{2}$ radians, note that $2\pi = \frac{4\pi}{2}$. Thus we want to add integral multiples of 4 to -9, until we get to a more familiar angle. Since $2(4\pi) + (-9\pi) = -\pi$, we have

$$2(2\pi) + (-\frac{9\pi}{2}) = -\frac{\pi}{2},$$

so that our angle of $-\frac{9\pi}{2}$ appears here on the unit circle.



The point on the unit circle corresponding to that angle is $(0, -1)$. Thus $\sin(-\frac{9\pi}{2}) = -1$.

Since $2,000,000\pi = 1,000,000(2\pi)$, the angle of $2,000,000\pi$ radians corresponds to revolving around the unit circle $1,000,000$ times. Even if you're dizzy, you still end up where you started: at the point $(1, 0)$. Thus $\cos(2,000,000\pi) = 1$.

c. By the Pythagorean theorem 5.11,

$$1 = \sin^2(\theta) + \cos^2(\theta) = \left(\frac{1}{3}\right)^2 + \cos^2(\theta),$$

so that

$$\cos(\theta) = \pm\sqrt{1 - \left(\frac{1}{3}\right)^2} = \pm\sqrt{1 - \frac{1}{9}} = \pm\sqrt{\frac{8}{9}} = \pm\frac{2\sqrt{2}}{3}.$$

The condition on θ will now tell us whether to choose the $+$ or the $-$. $\frac{5\pi}{2} = 2\pi + \frac{\pi}{2}$ and $\frac{7\pi}{2} = 2\pi + \frac{3\pi}{2}$; thus, since $\frac{5\pi}{2} \leq \theta \leq \frac{7\pi}{2}$, rotating θ radians from the positive x axis places us in the second or third quadrant. Cosine is negative there (see 5.10), thus we choose the negative square root:

$$\cos(\theta) = -\frac{2\sqrt{2}}{3}.$$

∎

5.13. Some famous angles.

$$\sin(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}.$$

$$\sin(\frac{\pi}{6}) = \cos(\frac{\pi}{3}) = \frac{1}{2}.$$

$$\sin(\frac{\pi}{3}) = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}.$$

5.14. Some more trig identities. You should be able to read the following from the unit circle definition 5.10.

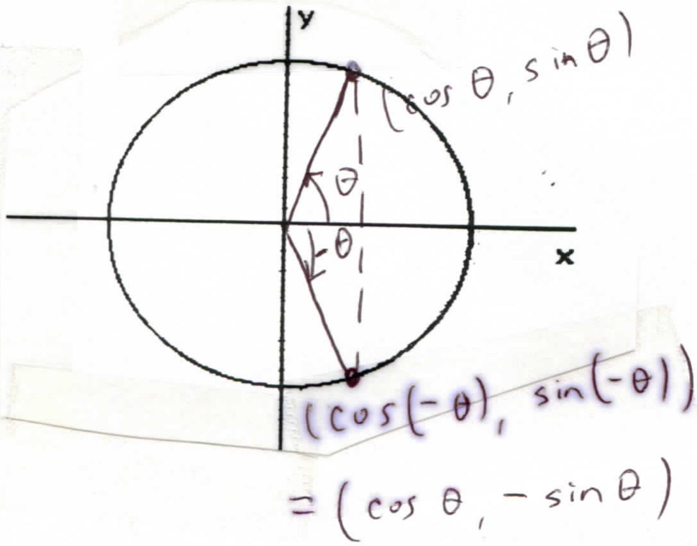
(a) $\sin(\theta + 2k\pi) = \sin(\theta), \cos(\theta + 2k\pi) = \cos(\theta)$, for any real θ, integer k.

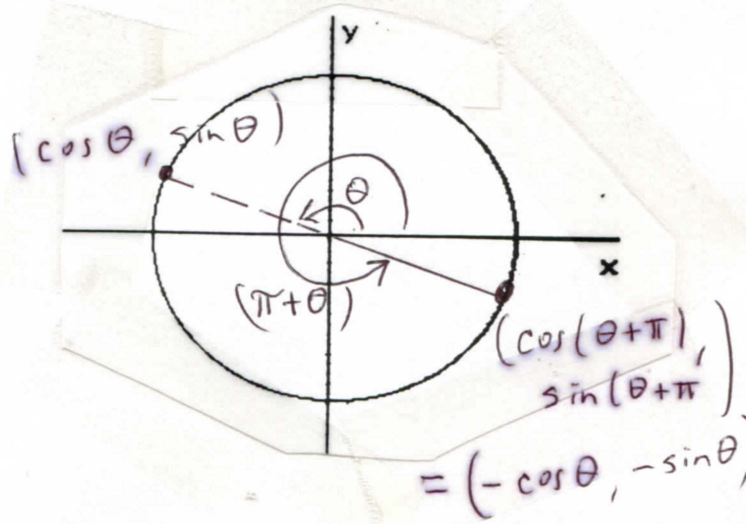
(b) $\sin(\theta \pm \pi) = -\sin(\theta), \cos(\theta \pm \pi) = -\cos(\theta)$, for any real θ.

(c) $\sin(-\theta) = -\sin(\theta)$, for any real θ.

(d) $\cos(-\theta) = \cos(\theta)$, for any real θ.

The identity in 5.14(b) is reflection through the origin; 5.14(c) and (d) is reflection through the x axis (see 3.3b).





The property in 5.14(a) is saying that sine and cosine are **periodic** functions; that is, they repeat themselves at regular intervals, something like a politician promising, every four years, to lower taxes.

5.15. Other trig functions. **Tangent** is defined by

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}.$$

Secant is defined by

$$\sec\theta \equiv \frac{1}{\cos\theta}.$$

Cosecant is defined by

$$\csc\theta \equiv \frac{1}{\sin\theta}.$$

Cotangent is defined by

$$\cot\theta \equiv \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}.$$

Examples 5.16. a. Find $\tan(\frac{\pi}{4})$.
b. Find $\csc(-\frac{\pi}{6})$.
c. Find $\cos(\frac{71\pi}{3})$.
d. Find $\tan(\frac{7\pi}{6})$.
e. Find $\sec(\frac{7\pi}{4})$.
f. Suppose $\tan(\theta) = -5$ and $\sin(\theta)$ is positive. What are $\sin(\theta)$ and $\cos(\theta)$?
g. Suppose $\sin(\theta) = \frac{2}{5}$. What is $\sin(7\pi - \theta)$?

Solutions. a. This is

$$\frac{\sin(\frac{\pi}{4})}{\cos(\frac{\pi}{4})} = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 1,$$

by 5.13.
b. This is

$$\frac{1}{\sin(-\frac{\pi}{6})} = \frac{1}{-\sin(\frac{\pi}{6})},$$

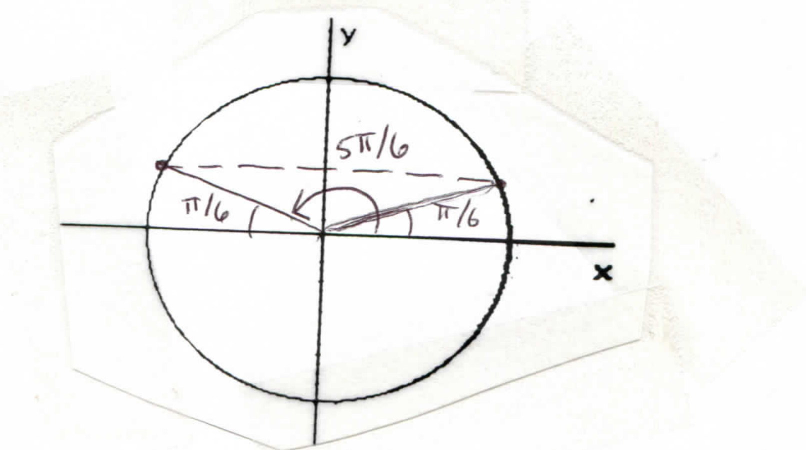
by 5.14(c), which equals $\frac{1}{-\frac{1}{2}} = -2$, by 5.13.
c. We need to subtract integral multiples of 2π to get to more familar territory. Since $2\pi = \frac{6\pi}{3}$, we want to subtract integral multiples of 6 from 71. Well, we could divide 6 into 71 with long division, to get

$$\frac{71}{6} = 11 + \frac{5}{6}.$$

By 5.14(a),

$$\cos(\frac{71\pi}{3}) = \cos(\frac{71\pi}{3} - 11(2\pi)) = \cos(\frac{71\pi}{3} - \frac{66\pi}{3}) = \cos(\frac{5\pi}{6}).$$

Now $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$, so we have located our angle:



By the symmetry in the drawing above, $\cos(\frac{5\pi}{6}) = -\cos(\frac{\pi}{6}) = -\frac{\sqrt{3}}{2}$, by 5.13.

OR, you could use 5.14(b) followed by (d):

$$\cos(\frac{5\pi}{6}) = -\cos(\frac{5\pi}{6} - \pi) = -\cos(-\frac{\pi}{6}) = -\cos(\frac{\pi}{6}).$$

d. Get to more familiar ground by subtracting π from $\frac{7\pi}{6}$ and using 5.14(c) and (d):

$$\tan(\frac{7\pi}{6}) = \frac{\sin(\frac{7\pi}{6})}{\cos(\frac{7\pi}{6})} = \frac{-\sin(\frac{7\pi}{6} - \pi)}{-\cos(\frac{7\pi}{6} - \pi)} = \frac{-\sin(\frac{\pi}{6})}{-\cos(\frac{\pi}{6})} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}.$$

e. Let's get an angle closer to 0 by subtracting 2π:

$$\sec(\frac{7\pi}{4}) = \frac{1}{\cos(\frac{7\pi}{4})} = \frac{1}{\cos(\frac{7\pi}{4} - 2\pi)} = \frac{1}{\cos(-\frac{\pi}{4})},$$

by 5.14(a); by 5.14(d), this equals

$$\frac{1}{\cos(\frac{\pi}{4})} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2},$$

by 5.13.

f. This is confusing enough to want to give things a name: Let

$$x \equiv \cos(\theta). \qquad (*)$$

We're told that

$$-5 = \tan(\theta) \equiv \frac{\sin(\theta)}{\cos(\theta)} = \frac{\sin(\theta)}{x},$$

thus

$$\sin(\theta) = -5x. \qquad (**)$$

We always have the Pythagorean theorem 5.11 to relate sin and cos, so plug in (*) and (**) into 5.11:

$$1 = \sin^2(\theta) + \cos^2(\theta) = x^2 + (-5x)^2 = x^2 + 25x^2 = 26x^2,$$

thus

$$\cos(\theta) = x = \pm\frac{1}{\sqrt{26}}.$$

Since $\tan(\theta)$ is negative and we're told that $\sin(\theta)$ is positive, it follows from the definition of tangent (see 5.15) that $\cos(\theta)$ must be negative. So we choose the negative square root:

$$\cos(\theta) = -\frac{1}{\sqrt{26}},$$

and from (**),

$$\sin(\theta) = \frac{5}{\sqrt{26}}.$$

g. Use 5.14 to move from the angle of $(7\pi - \theta)$ to θ. By 5.14(a),

$$\sin(7\pi - \theta) = \sin(7\pi - \theta - 3(2\pi)) = \sin(\pi - \theta).$$

By 5.14(b),

$$\sin(\pi - \theta) = -\sin(-\theta).$$

By 5.14(c),

$$-\sin(-\theta) = \sin(\theta).$$

Thus $\sin(7\pi - \theta) = \sin(\theta) = \frac{2}{5}$. ∎

5.17. Sum-of-angles formulas. For any real α, β,

(a)

$$\cos(\alpha + \beta) = (\cos(\alpha))(\cos(\beta)) - (\sin(\alpha))(\sin(\beta)),$$

(b)

$$\sin(\alpha + \beta) = (\cos(\alpha))(\sin(\beta)) + (\sin(\alpha))(\cos(\beta)),$$

and

(c)

$$\tan(\alpha + \beta) = \frac{[\tan(\beta) + \tan(\alpha)]}{[1 - \tan(\alpha)\tan(\beta)]}.$$

Examples 5.18. a. Suppose $\sin\alpha = \frac{1}{3}$ and $\sin\beta = \frac{2}{5}$. Find $\sin(\alpha + \beta)$ and $\sin(\alpha - \beta)$, if $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ and $\frac{\pi}{2} < \beta < \frac{3\pi}{2}$.

b. Suppose $\tan\alpha = 20$ and $\tan\beta = 2$. Find $\tan(\alpha + \beta)$ and $\tan(\alpha - \beta)$.

c. Find $\cos(\frac{\pi}{12})$.

d. Find $\tan(\frac{7\pi}{12})$.

Solutions. a. By 5.17(b),

$$\sin(\alpha + \beta) = \frac{2}{5}\cos(\alpha) + \frac{1}{3}\cos(\beta).$$

We can use the Pythagorean theorem to get the cosines. By 5.11

$$1 = (\cos(\alpha))^2 + \left(\frac{1}{3}\right)^2 \quad \text{and} \quad 1 = (\cos(\beta))^2 + \left(\frac{2}{5}\right)^2,$$

so that we may solve for $\cos(\alpha)$ and $\cos(\beta)$:

$$\cos(\alpha) = \pm\sqrt{\frac{8}{9}} = \pm\frac{2\sqrt{2}}{3}, \quad \cos(\beta) = \pm\sqrt{\frac{21}{25}} = \pm\frac{\sqrt{21}}{5}.$$

The conditions on α and β are needed to choose between $+$ and $-$. The condition on α places $(\cos(\alpha), \sin(\alpha))$ in the first or fourth quadrant, so that $\cos(\alpha)$ is positive (see 5.10); the condition on β places $(\cos(\beta), \sin(\beta))$ in the second or third quadrant, so that $\cos(\beta)$ is negative, so that we finally have

$$\sin(\alpha+\beta) = \left(\frac{2}{5}\right)\left(\frac{2\sqrt{2}}{3}\right) + \left(\frac{1}{3}\right)\left(-\frac{\sqrt{21}}{5}\right).$$

Out of decency, let's simplify this:

$$\sin(\alpha+\beta) = \frac{(4\sqrt{2}-\sqrt{21})}{15}.$$

b. By 5.17(c),

$$\tan(\alpha+\beta) = \frac{(2+20)}{(1-(20)(2))} = -\frac{22}{39},$$

and, by 5.14(c) and (d),

$$\begin{aligned}\tan(\alpha-\beta) = \tan(\alpha+(-\beta)) &= \frac{[\tan(-\beta)+\tan(\alpha)]}{[1-\tan(\alpha)\tan(-\beta)]} \\ &= \frac{[-\tan(\beta)+\tan(\alpha)]}{[1+\tan(\alpha)\tan(\beta)]} = \frac{(-2+20)}{(1+(20)(2))} = \frac{18}{41}.\end{aligned}$$

c. Since $\frac{1}{12} = \frac{4-3}{12} = \frac{1}{3} - \frac{1}{4}$, we can write

$$\begin{aligned}\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{3}+(-\frac{\pi}{4})\right) &= \left(\cos(\frac{\pi}{3})\right)\left(\cos(-\frac{\pi}{4})\right) - \left(\sin(\frac{\pi}{3})\right)\left(\sin(-\frac{\pi}{4})\right) \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{\sqrt{2}}\right) = \frac{(1+\sqrt{3})}{2\sqrt{2}},\end{aligned}$$

by 5.17(a).

d. Since $\frac{7}{12} = \frac{9-2}{12} = \frac{3}{4} - \frac{1}{6}$, we will use 5.17(c) as follows:

$$\begin{aligned}\tan\left(\frac{7\pi}{12}\right) = \tan\left(\frac{3\pi}{4}+(-\frac{\pi}{6})\right) &= \frac{\left[\tan(-\frac{\pi}{6})+\tan(\frac{3\pi}{4})\right]}{\left[1-\tan(\frac{3\pi}{4})\tan(-\frac{\pi}{6})\right]} \\ &= \frac{\left(-\frac{1}{\sqrt{3}}+(-1)\right)}{\left[1-(-1)(-\frac{1}{\sqrt{3}})\right]} = \frac{(1+\sqrt{3})}{(1-\sqrt{3})}.\end{aligned}$$

∎

Corollary 5.19. *For any real θ,*

$$\sin(\theta + \frac{\pi}{2}) = \cos(\theta), \quad \cos(\theta + \frac{\pi}{2}) = -\sin(\theta),$$

$$\sin(\theta - \frac{\pi}{2}) = -\cos(\theta), \quad \cos(\theta - \frac{\pi}{2}) = \sin(\theta).$$

Proof: Use the Sum-of-angles formulas:

$$\sin(\theta + \frac{\pi}{2}) = \sin(\theta)\cos(\frac{\pi}{2}) + \sin(\frac{\pi}{2})\cos(\theta) = \cos(\theta),$$

and

$$\cos(\theta + \frac{\pi}{2}) = \cos(\theta)\cos(\frac{\pi}{2}) - \sin(\theta)\sin(\frac{\pi}{2}) = -\sin(\theta),$$

since $\sin(\frac{\pi}{2}) = 1$ and $\cos(\frac{\pi}{2}) = 0$.

I'll leave it to you to do the same argument with $(\theta - \frac{\pi}{2})$. ∎

Corollary 5.19 can also be shown by drawing right triangles on the unit circle and performing geometry.

5.20. Half-angle formulas: For any real θ,

(a)

$$\sin(\theta) = \pm\sqrt{\frac{1}{2}(1 - \cos(2\theta))},$$

(b)

$$\cos(\theta) = \pm\sqrt{\frac{1}{2}(1 + \cos(2\theta))},$$

and

(c)

$$\tan(\theta) = \frac{\sin(2\theta)}{1 + \cos(2\theta)}.$$

Examples 5.21. a. Find $\sin(\frac{\pi}{12})$.

b. Find $\tan(-\frac{5\pi}{8})$.

c. If $\tan(\theta) = 2$, and $\pi < \theta < \frac{3\pi}{2}$, find sin and cos of $\frac{\theta}{2}, \theta$ and 2θ.

Solutions. a. Apply 5.20(a), with $\theta \equiv \frac{\pi}{12}$, so that $2\theta = \frac{\pi}{6}$:

$$\sin\left(\frac{\pi}{12}\right) = \pm\sqrt{\frac{1}{2}\left(1 - \cos(\frac{\pi}{6})\right)}$$

$$= \pm\sqrt{\frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right)} = \pm\sqrt{\frac{(2-\sqrt{3})}{4}} = \pm\frac{\sqrt{(2-\sqrt{3})}}{2}.$$

Since $0 < \frac{\pi}{12} < \frac{\pi}{2}$, $(\sin(\frac{\pi}{12}), \cos(\frac{\pi}{12}))$ is in the first quadrant, so that $\sin(\frac{\pi}{12})$ is positive. Thus we choose the positive square root:

$$\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{(2-\sqrt{3})}}{2}.$$

b. Let $\theta \equiv -\frac{5\pi}{8}$, so that $2\theta = -\frac{5\pi}{4}$, and apply 5.20(c), followed by a few applications of 5.14:

$$\begin{aligned}\tan\left(-\frac{5\pi}{8}\right) &= \frac{\sin(-\frac{5\pi}{4})}{1+\cos(-\frac{5\pi}{4})} \\ &= \frac{\sin(-\pi-\frac{\pi}{4})}{1+\cos(-\pi-\frac{\pi}{4})} = \frac{-\sin(-\frac{\pi}{4})}{1-\cos(-\frac{\pi}{4})} \\ &= \frac{\sin(\frac{\pi}{4})}{1-\cos(\frac{\pi}{4})} = \frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}-1}.\end{aligned}$$

c. We'd better start with sin and cos of θ, since both the Sum-of-angles and Half-angle formulas depend on them.

We reason as in Example 5.16f. Let $x \equiv \cos(\theta)$. By definition of tan,

$$2 = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\sin(\theta)}{x},$$

so that

$$\sin(\theta) = 2x.$$

Then by the Pythagorean theorem 5.11,

$$1 = \sin^2(\theta) + \cos^2(\theta) = (2x)^2 + x^2 = 5x^2,$$

thus

$$\cos(\theta) = x = -\sqrt{\frac{1}{5}} = -\frac{1}{\sqrt{5}};$$

I chose the negative square root because $\pi < \theta < \frac{3\pi}{2}$, so that $(\cos(\theta), \sin(\theta))$ is in the third quadrant. Thus

$$\sin(\theta) = -\frac{2}{\sqrt{5}}.$$

Now we can apply the Half-angle formulas 5.20(a) and (b), with θ replaced by $\frac{\theta}{2}$:

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1}{2}(1-\cos(\theta))} = \sqrt{\frac{1}{2}\left(1-\left(-\frac{1}{\sqrt{5}}\right)\right)} = \sqrt{\frac{1}{2}\left(1+\left(\frac{1}{\sqrt{5}}\right)\right)};$$

I chose the positive square root because $\frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}$. Also

$$\cos\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1}{2}(1+\cos(\theta))} = -\sqrt{\frac{1}{2}\left(1-\frac{1}{\sqrt{5}}\right)};$$

here I chose the negative square root for cosine because $\frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}$.

For $\sin(2\theta)$ and $\cos(2\theta)$ use the Sum-of-angles formulas 5.17 with $\alpha = \beta = \theta$:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\left(-\frac{2}{\sqrt{5}}\right)\left(-\frac{1}{\sqrt{5}}\right) = \frac{4}{5},$$

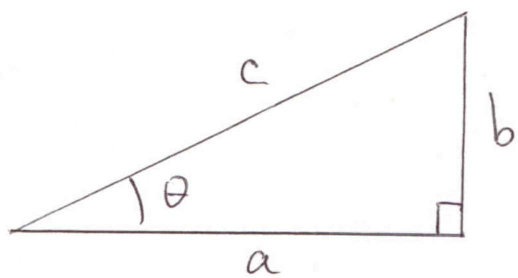
and

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \left(-\frac{1}{\sqrt{5}}\right)^2 - \left(-\frac{2}{\sqrt{5}}\right)^2 = -\frac{3}{5}.$$

∎

5.22. Trig functions on a right triangle. An angle θ is **acute** if $0 \le \theta < \frac{\pi}{2}$ (measured in radians).

For an acute angle θ, we may express $\sin\theta, \cos\theta$, etc., in terms of the sides of a right triangle.



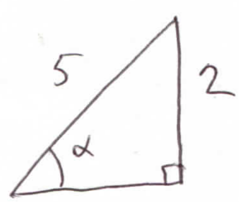
In the triangle above,

$$\sin\theta = \frac{b}{c} \text{ ("opposite over hypotenuse")},$$

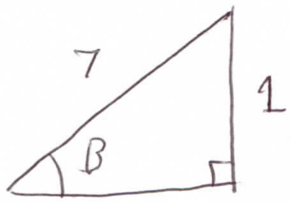
$$\cos\theta = \frac{a}{c} \text{ ("adjacent over hypotenuse")},$$

$$\tan\theta = \frac{b}{a} \text{ ("opposite over adjacent")}.$$

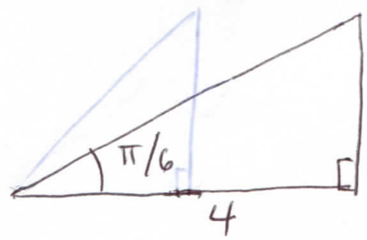
Examples 5.23. a. Find $\sin(\alpha)$ in the following triangle.



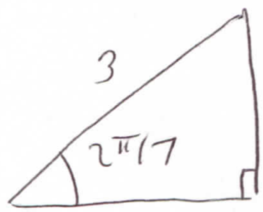
b. Find $\tan(\beta)$ in the following triangle.



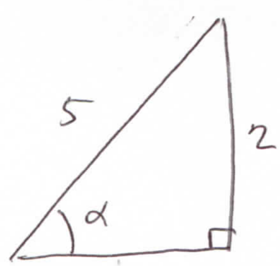
c. Find the lengths of all sides of the following triangle.



d. Find the lengths of all sides of the following triangle.



e. Find sin and cos of α, $\frac{\alpha}{2}$ and 2α, from the following triangle.



Solutions. a. Opposite divided by hypotenuse is $\frac{2}{5}$.

b. Let x be the (length of the) adjacent side to β. By the Pythagorean theorem, $x^2 + 1^2 = 7^2$, thus $x = \sqrt{48} = 4\sqrt{3}$, so by 5.22, $\tan(\alpha) = \frac{1}{4\sqrt{3}}$.

c. Let's write c for the (length of the) hypotenuse and b for the (length of the) side opposite to $\frac{\pi}{6}$. Then by 5.22 and 5.13,

$$\frac{\sqrt{3}}{2} = \cos(\frac{\pi}{6}) = \frac{4}{c},$$

so, solving for c, the hypotenuse has length $\frac{8}{\sqrt{3}}$.

Also

$$\frac{1}{2} = \sin(\frac{\pi}{6}) = \frac{b}{c} = \frac{b}{\frac{8}{\sqrt{3}}}.$$

so the length of the side opposite to $\frac{\pi}{6}$ is $\frac{4}{\sqrt{3}}$.

d. This is the same strategy as part c: let a be the (length of the) side adjacent to $\frac{2\pi}{7}$, b be the (length of the) side opposite to $\frac{2\pi}{7}$. Then by 5.22,

$$\sin(\frac{2\pi}{7}) = \frac{b}{3},$$

so the length of the side opposite to $\frac{2\pi}{7}$ is $3\sin(\frac{2\pi}{7})$. Likewise,

$$\cos(\frac{2\pi}{7}) = \frac{a}{3},$$

so the length of the side adjacent to $\frac{2\pi}{7}$ is $3\cos(\frac{2\pi}{7})$.

These answers cannot be simpified. A calculator would give you only an *approximation* (close, but not equal) to the answer.

e. The side of the triangle adjacent to α has length $\sqrt{5^2 - 2^2} = \sqrt{21}$, by the Pythagorean theorem. Now we can read off sin and cos of α from the triangle (see 5.22):

$$\sin(\alpha) = \frac{2}{5} \quad \text{and} \quad \cos(\alpha) = \frac{\sqrt{21}}{5}.$$

We use 5.22 and 5.17 to get sin and cos of $\frac{\alpha}{2}$ and 2α; note that sin and cos of $\frac{\alpha}{2}$ will be positive, since $0 < \alpha < \frac{\pi}{2}$.

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1}{2}(1 - \cos(\alpha))} = \sqrt{\frac{1}{2}\left(1 - \frac{\sqrt{21}}{5}\right)},$$

and

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1}{2}(1 + \cos(\alpha))} = \sqrt{\frac{1}{2}\left(1 + \frac{\sqrt{21}}{5}\right)}.$$

By 5.17, with $\beta = \alpha$,

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) = 2\left(\frac{2}{5}\right)\left(\frac{\sqrt{21}}{5}\right) = \frac{4\sqrt{21}}{25},$$

and

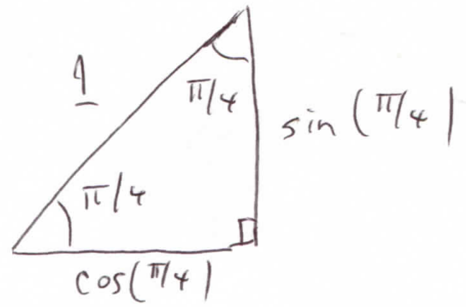
$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = \left(\frac{\sqrt{21}}{5}\right)^2 - \left(\frac{2}{5}\right)^2 = \frac{17}{25}.$$

∎

5.24. Derivation of famous angles in 5.13. For $\sin(\frac{\pi}{4})$, use the Pythagorean theorem and the fact that $\sin(\frac{\pi}{4})$ must equal $\cos(\frac{\pi}{4})$, by 5.22:

$$1 = \sin^2(\frac{\pi}{4}) + \cos^2(\frac{\pi}{4}) = 2\sin^2(\frac{\pi}{4}),$$

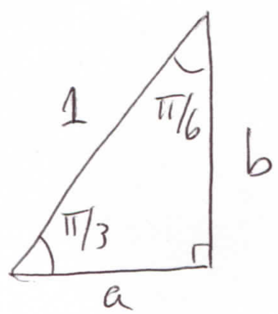
so that solving for $\sin(\frac{\pi}{4})$ gives us $\frac{1}{\sqrt{2}}$.



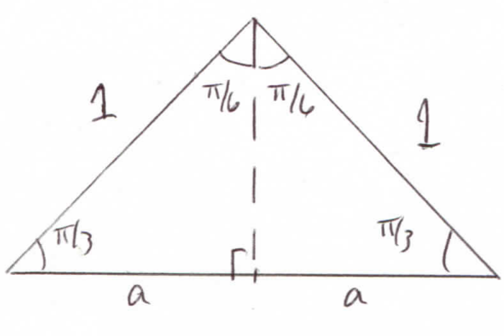
For the angles $\frac{\pi}{6}$ and $\frac{\pi}{3}$, first note that, by 5.22,

$$\sin(\frac{\pi}{6}) = \cos(\frac{\pi}{3}) \quad \text{and} \quad \sin(\frac{\pi}{3}) = \cos(\frac{\pi}{6}).$$

Let's write a for $\sin(\frac{\pi}{6})$, b for $\sin(\frac{\pi}{3})$.



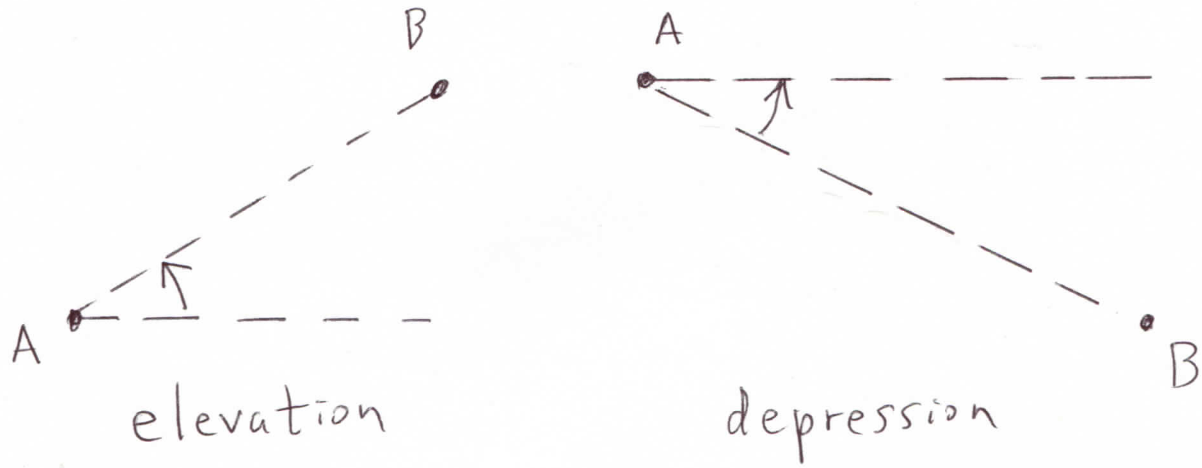
Reflect that right triangle through its right side, to make an equilateral triangle (since the angles are equal, the sides must be equal).



Since it's equilateral, $2a = 1$ or $a = \frac{1}{2}$; now apply the Pythagorean theorem $a^2 + b^2 = 1$ and solve for b, to get $b = \frac{\sqrt{3}}{2}$.

5.25. If A and B are two points, with B higher above the ground than A, then the **angle of elevation** of B from A is the angle between a horizontal line (parallel to the ground) from A and a line from A to B (called the *line of sight*).

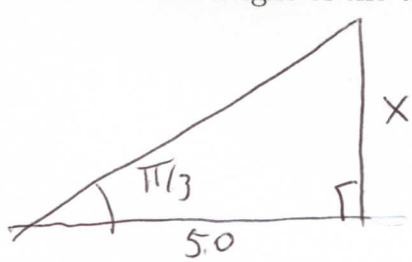
If B is below A, this angle is called the **angle of depression** of B from A.



Examples 5.26. a. Suppose you are standing on the ground 50 feet from the base of a tree, and the angle of elevation of the top of the tree from your feet is $\frac{\pi}{3}$. How tall is the tree?

b. You see a UFO hovering above the ground. At your house, the angle of elevation of the UFO from the ground is $\frac{\pi}{4}$. At your neighbor's house 100 yards along the ground in a direction away from the UFO the angle of elevation is $\frac{\pi}{6}$. How high above the ground is the UFO?

Solutions. a. Let x be the desired height of the tree. Here's the picture.

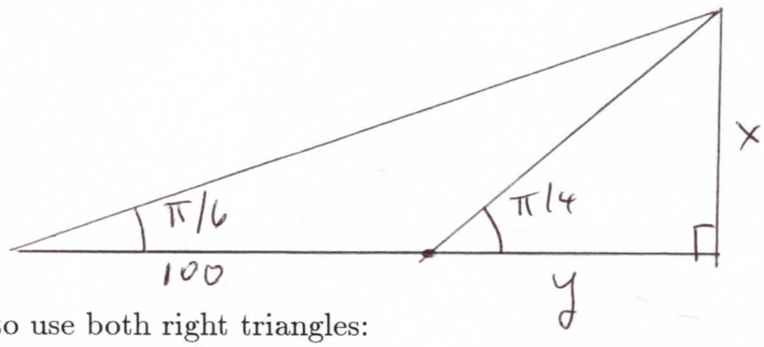


By 5.22 and 5.13,

$$\sqrt{3} = \tan(\frac{\pi}{3}) = \frac{x}{50},$$

so x equals $50\sqrt{3}$ feet.

b. Let x be the height above the ground of the UFO and let y be the distance between your house and the point directly below the UFO. This is the picture.



We need to use both right triangles:

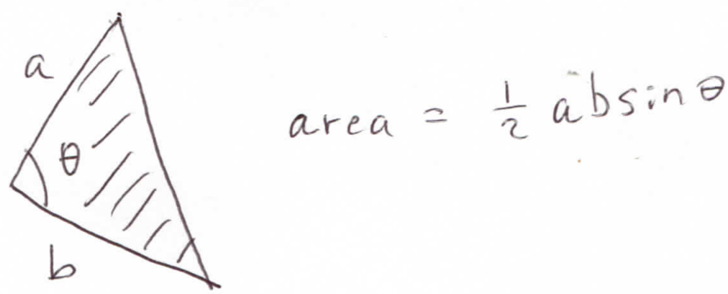
$$1 = \tan\left(\frac{\pi}{4}\right) = \frac{x}{y} \quad \text{and} \quad \frac{1}{\sqrt{3}} = \tan\left(\frac{\pi}{6}\right) = \frac{x}{y+100}.$$

We can solve these equations for x. By the first equation, $x = y$, so the second equation becomes

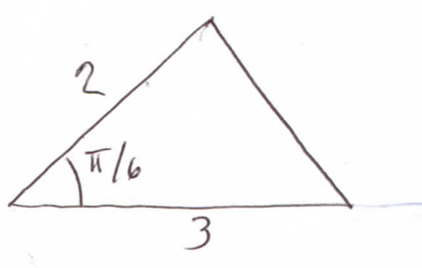
$$\frac{1}{\sqrt{3}} = \frac{x}{x+100},$$

so that $x + 100 = \sqrt{3}x$, and x equals $\frac{100}{(\sqrt{3}-1)}$ feet. ∎

5.27. Area of triangle. If the angle between two sides of a triangle, of lengths a and b, is θ, then the area of the triangle is $\frac{1}{2}ab\sin(\theta)$.



Example 5.28. Find the area of the following triangle, if the units are meters.
$(a = 2, b = 3, \theta = \frac{\pi}{6})$



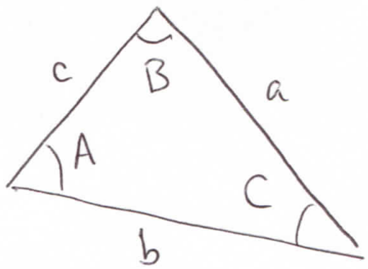
Solution. Here $a = 2, b = 3$ and $\theta = \frac{\pi}{6}$, so by 5.27, the area is

$$\frac{1}{2}(2 \text{ meters })(3 \text{ meters })\sin(\frac{\pi}{6}) = \frac{1}{2}(2 \text{ meters })(3 \text{ meters })\frac{1}{2} = \frac{3}{2} \text{ meters squared.}$$

▮

5.29. Law of sines. In a given triangle, the ratio of the sine of an angle to the length of the side opposite that angle is constant.

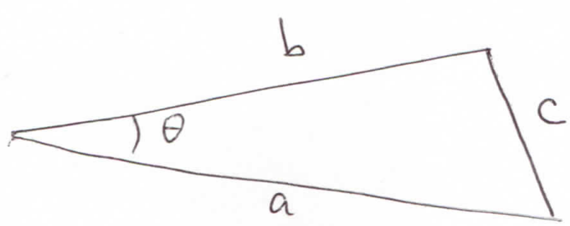
More explicitly: Suppose a triangle has sides of lengths a, b and c, the angle opposite the side of length a measures A radians, the angle opposite the side of length b measures B radians and the angle opposite the side of length c measures C radians.



Then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

5.30. Law of cosines. Suppose a triangle has sides of lengths a, b and c, and the angle opposite the side of length c measures θ radians.

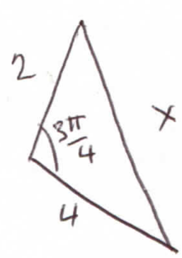


Then

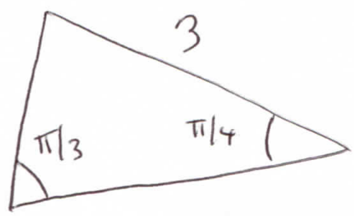
$$c^2 = a^2 + b^2 - 2ab\cos(\theta).$$

Note that the Pythagorean theorem is a special case of the Law of Cosines (when $\theta = \frac{\pi}{2}$).

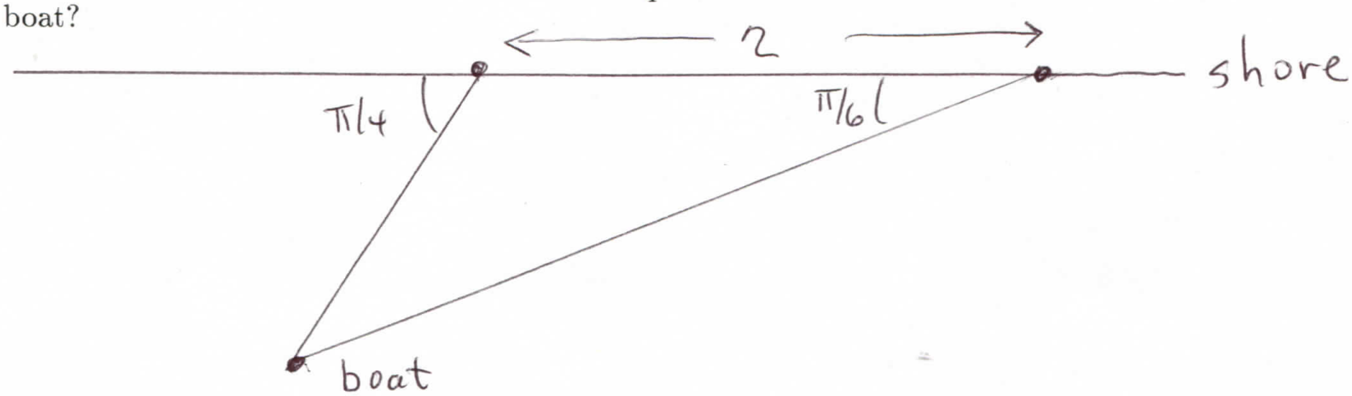
Examples 5.31. a. Find x, the length of the indicated side. All dimensions are in meters.



b. Find the lengths of all the sides of the following triangle, where the units are inches.



c. You are standing on a straight shore staring at an anchored boat in the ocean. At Point Drizzle, the angle between the shore and a line from you to the boat is $\frac{\pi}{6}$. When you walk two miles down the shore from Point Drizzle, the angle between the shore and a line from you to the boat is $\frac{\pi}{4}$. How far from Point Drizzle is the boat?



d. In c., how far from the shore is the boat? That is, what is the distance to the point on the shore closest to the boat?

e. In c., how far from Point Drizzle must you walk to reach the point on the shore closest to the boat?

Solutions. a. Use the Law of Cosines, with $a = 2, b = 4, \theta = \frac{3\pi}{4}, c = x$:

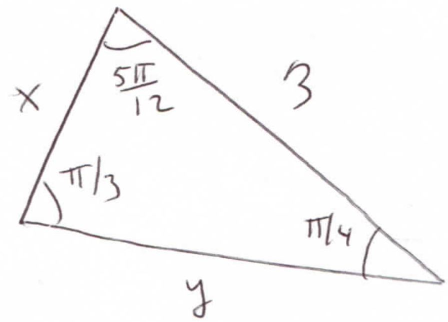
$$x^2 = 2^2 + 4^2 - 2(2)(4)\cos\left(\frac{3\pi}{4}\right) = 4 + 16 - 16\left(-\frac{1}{\sqrt{2}}\right) = 20 + \frac{16}{\sqrt{2}};$$

thus, the length we want is $x = \sqrt{20 + \frac{16}{\sqrt{2}}} = 2\sqrt{5 + \frac{4}{\sqrt{2}}}$ meters.

b. First, let's get the remaining angle of the triangle. Since the sum of angles in a triangle is π radians, the missing angle measures

$$(\pi - \frac{\pi}{3} - \frac{\pi}{4}) = \frac{5\pi}{12} \text{ radians .}$$

Let's also label our unknown sides.



Now use the Law of Sines:

$$\frac{1}{2\sqrt{3}} = \frac{\sin(\frac{\pi}{3})}{3} = \frac{\sin(\frac{\pi}{4})}{x} = \frac{\sin(\frac{5\pi}{12})}{y},$$

thus $x = (2\sqrt{3})\sin(\frac{\pi}{4}) = \sqrt{6}$ inches.

We can use Sum-of-angles 5.17 to find $\sin(\frac{5\pi}{12})$. Since $5 = 8 - 3$, we have

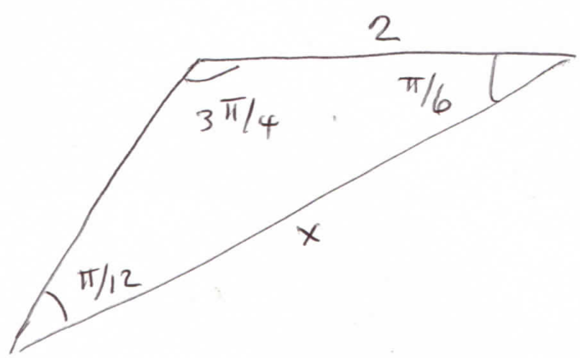
$$\frac{5\pi}{12} = \frac{8\pi}{12} - \frac{3\pi}{12} = \frac{2\pi}{3} - \frac{\pi}{4},$$

thus

$$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{2\pi}{3} + \left(-\frac{\pi}{4}\right)\right) = \sin\left(\frac{2\pi}{3}\right)\cos\left(-\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right)\cos\left(\frac{2\pi}{3}\right)$$
$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{2}\right) = \frac{(\sqrt{3}+1)}{2\sqrt{2}},$$

so that $y = 2\sqrt{3}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) = \frac{(3+\sqrt{3})}{\sqrt{2}}$ inches.

c. Let x be the distance from Point Drizzle to the boat. Here's the relevant triangle.



By the Law of Sines,

$$\frac{\sin(\frac{3\pi}{4})}{x} = \frac{\sin(\frac{\pi}{12})}{2}. \qquad (*)$$

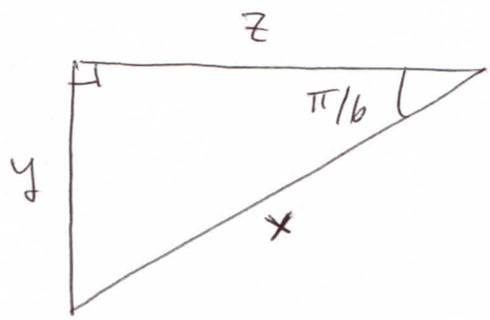
As in Example 5.18, we may use Sum-of-angles 5.17 to calculate

$$\sin(\frac{\pi}{12}) = \sin(\frac{\pi}{3} + (-\frac{\pi}{4})) = \sin(\frac{\pi}{3})\cos(-\frac{\pi}{4}) + \sin(-\frac{\pi}{4})\cos(\frac{\pi}{3})$$
$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{(\sqrt{3}-1)}{2\sqrt{2}}.$$

Thus (*) gives us

$$x = 2\left(\frac{\sin(\frac{3\pi}{4})}{\sin(\frac{\pi}{12})}\right) = 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{2\sqrt{2}}{(\sqrt{3}-1)}\right) = \frac{4}{(\sqrt{3}-1)} \text{ miles .}$$

d. The line from the boat to the point on the shore closest to the boat is perpendicular to the shore. Here is a drawing we'll need for both d. and e.; I've put y for the distance to the shore, z for the distance from Point Drizzle to the point on the shore closest to the boat, and x is as in part c., the distance from Point Drizzle to the boat.



In this picture,

$$\frac{y}{x} = \sin(\frac{\pi}{6}) = \frac{1}{2},$$

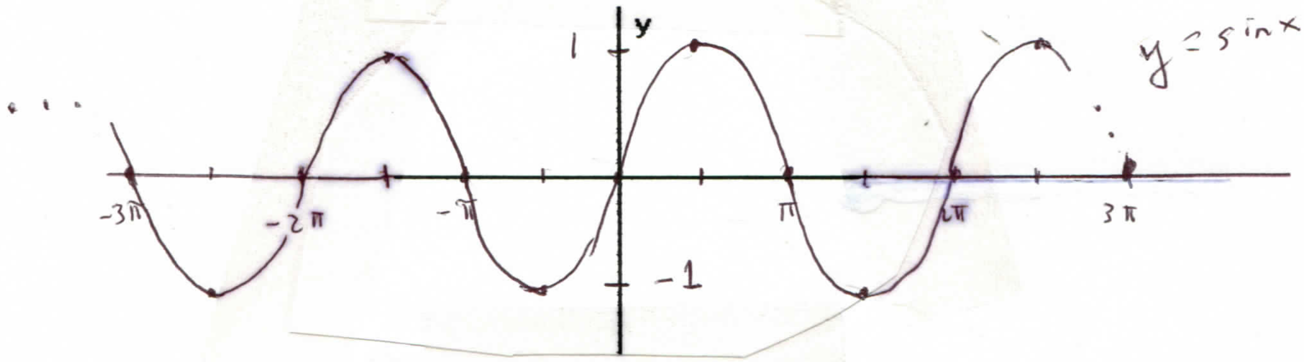
thus $y = \frac{1}{2}x = \frac{2}{(\sqrt{3}-1)}$ miles (we got x in part c.).

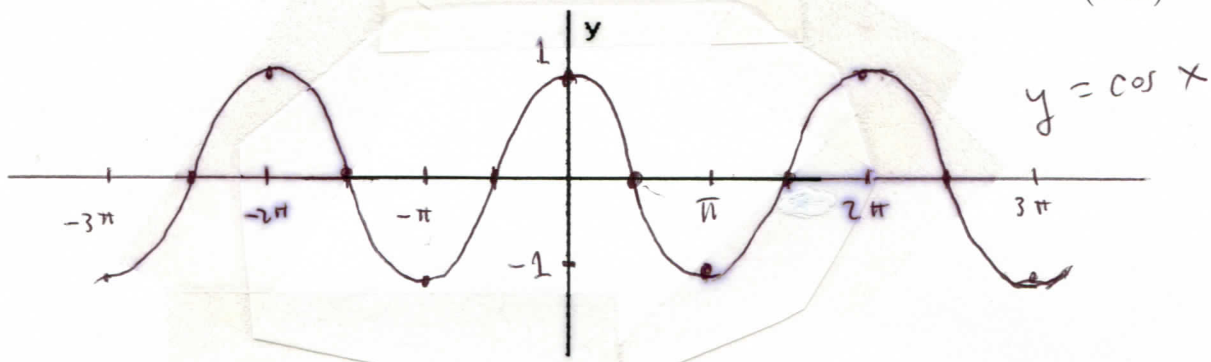
e. Use the same triangle that we looked at in the solution to d.; we want the remaining side, call it z, besides x and y. I'll use cosine:

$$\frac{z}{x} = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2},$$

so $z = \frac{2\sqrt{3}}{(\sqrt{3}-1)}$ miles. ∎

Using 5.13, 5.14 and the periodicity of sin and cos, we may graph $y = \sin x$ and $y = \cos x$.





(5.32)

5.33. Stretching and shifting the graphs of sine and cosine. We also want to make some distortions of sin and cos graphs. Consider, for A, α and β real numbers, $\alpha > 0$,

$$f(x) = A\sin(\alpha x - \beta) + c \quad \text{or} \quad f(x) = A\cos(\alpha x - \beta) + c. \tag{5.34}$$

(1) $|A|$ is called the **amplitude**. This is half the distance between the maximum and minimum values of the function.

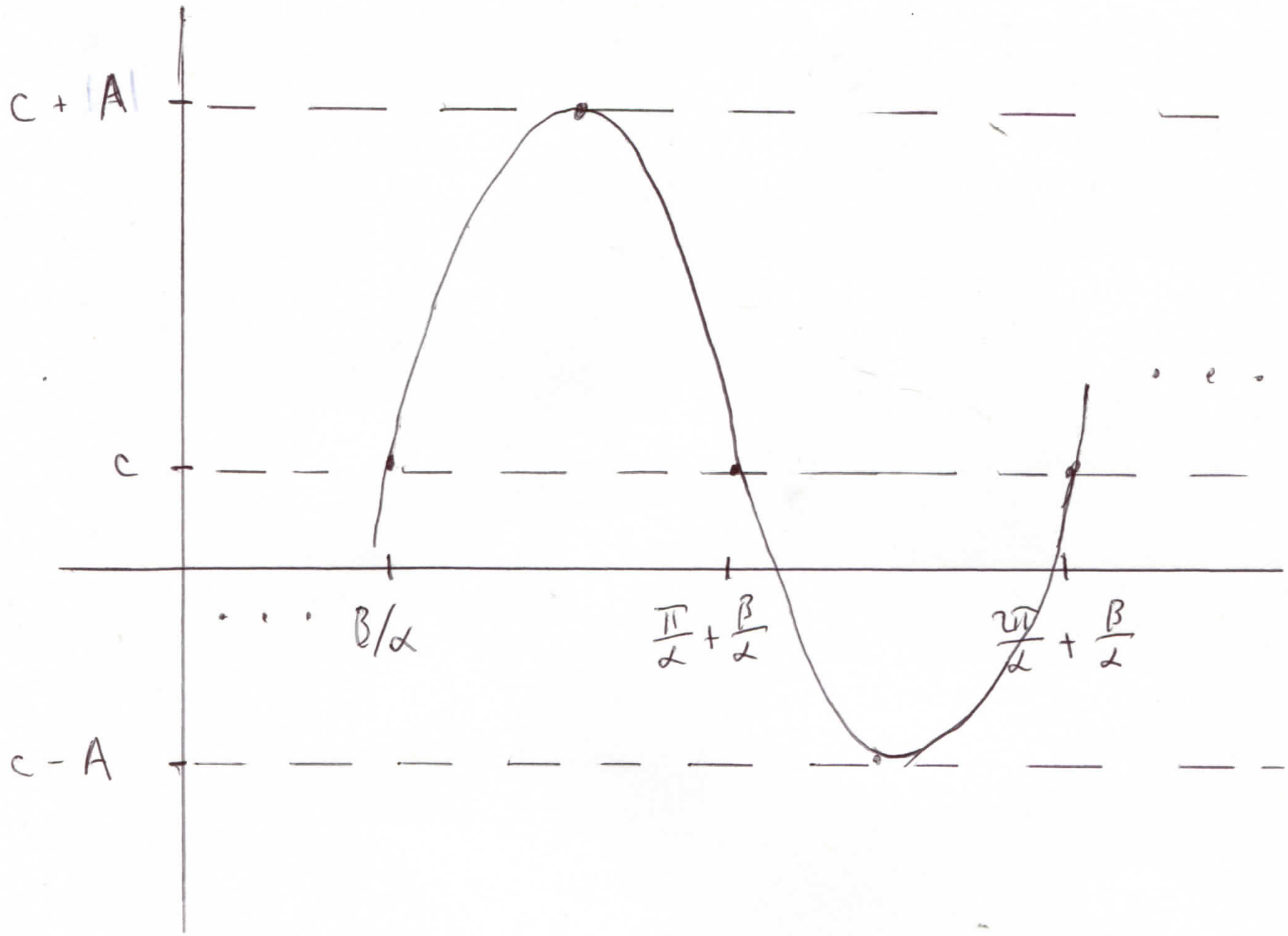
(2) $\frac{2\pi}{\alpha}$ is the **period** or wavelength. This is the minimum time the function needs to repeat itself, that is, the smallest p so that

$$f(x + p) = f(x) \quad \text{for all real } x.$$

(3) $\frac{\alpha}{2\pi}$ is the **frequency**. This measures how rapidly the function repeats. Note that frequency is $\frac{1}{\text{period}}$.

(4) $\frac{\beta}{\alpha}$ is the **phase shift**. (If $g(x) \equiv A\sin(\alpha x) + c$, then $f(x) = g(x - \frac{\beta}{\alpha})$; the phase shift is a horizontal translation, as in 3.3a.)

(5) c is a vertical translation, as in 3.3a.



$$y = A\sin(\alpha x - \beta) + c$$

$(A > 0)$

Sound, light and other electromagnetic waves, the number of hours of daylight in a day—many phenomena that repeat themselves are described by (5.34). Some philosophies have the entire universe being periodic as a function of time.

For a sound wave, the amplitude corresponds to the volume, the frequency corresponds to the pitch. A high note has a small period and a large frequency, a low note has a large period and a small frequency. An octave corresponds to doubling the frequency of vibration.

For visible light (wavelength of 10^{-6} meters), the frequency corresponds to the color: in order of increasing frequency (decreasing wavelength), the colors are

red–orange–yellow–green–blue–purple.

The electromagnetic waves with the next highest frequencies after purple are called ultraviolet radiation; this gives you a suntan (and wrinkles, they say). Next come X-rays, then gamma rays (wavelength of 10^{-10} meters or less); the higher the frequency the more destructive potential the waves have. At the other extreme, radio waves are electromagnetic waves with periods between 10^{-3} and 10^{8} meters.

An important subject in math, science and engineering is *Fourier analysis;* this involves writing almost any function as a sum of functions as in (5.34).

Examples 5.35. For each of the following, graph the function for one period and find the amplitude, period, frequency, and phase shift of the graph. Find where the graph is increasing, where it is decreasing, and find the maximum and minimum values.

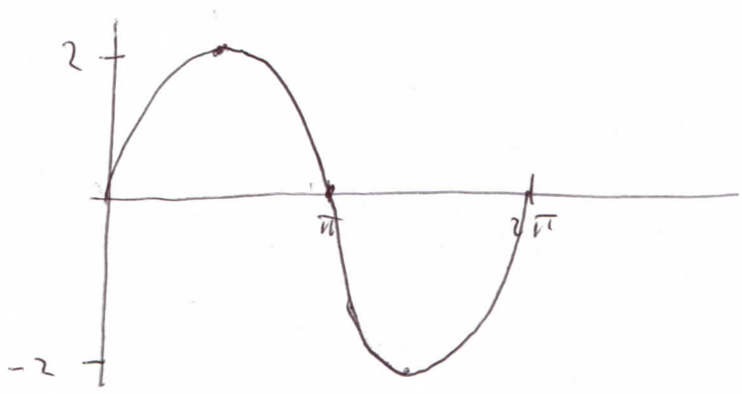
a. $f(x) = 2\sin(3x + \frac{\pi}{4})$.

b. $f(x) = -3\cos(\frac{x}{2}) + 3$.

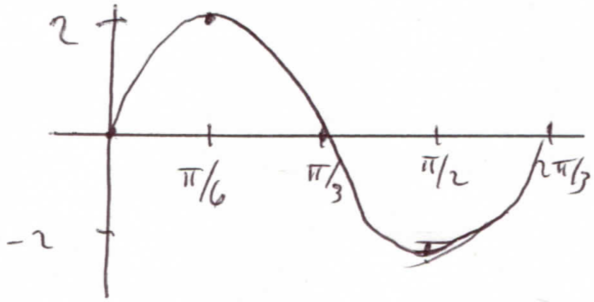
Solutions. We can read off the desired numbers by comparing the definition of the function to (5.34).

a. Here $A = 2, \alpha = 3$ and $\beta = -\frac{\pi}{4}$. So the amplitude of the graph is 2, the period is $\frac{2\pi}{3}$, the frequency is $\frac{3}{2\pi}$ cycles per unit time and the phase shift is $-\frac{\pi}{12}$.

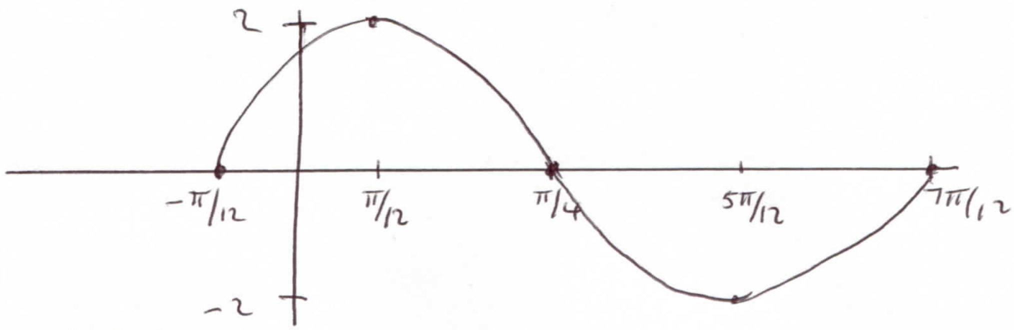
I recommend drawing the graph in stages, as in 3.4. Start with one period, say $0 \leq x \leq 2\pi$, of the graph of $y = \sin x$ in (5.32); first put on the amplitude, so that the range goes from -2 to 2 instead of -1 to 1:



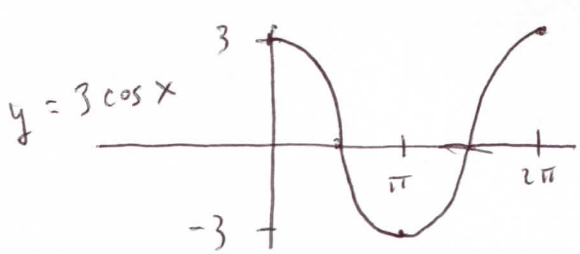
Now replace x with $3x$; this means, in your graph, you divide each important x-value by 3: $\frac{\pi}{2}$ is replaced by $\frac{\pi}{6}$, π is replaced by $\frac{\pi}{3}$, etc.

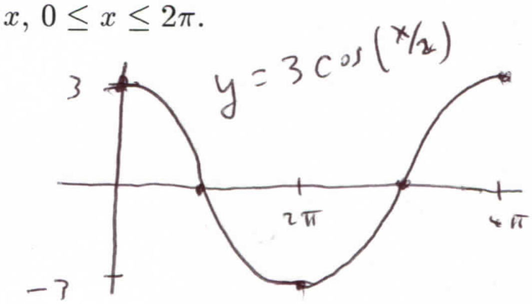


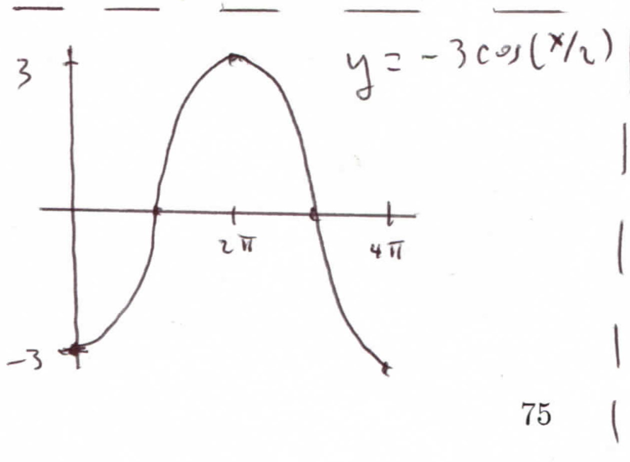
Now add on the phase shift; that's a horizontal translation of $-\frac{\pi}{12}$ units to the right, which means $\frac{\pi}{12}$ units to the left.

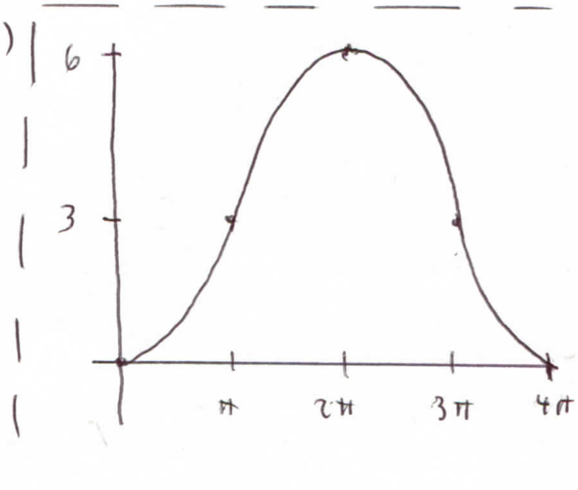


We can read off where the graph is increasing or decreasing, and what its maximum and minimum are, by staring at the graph: the maximum is 2, the minimum is -2, the graph is increasing on $(-\frac{\pi}{12}, \frac{\pi}{12})$ and $(\frac{5\pi}{12}, \frac{7\pi}{12})$, and is decreasing on $(\frac{\pi}{12}, \frac{5\pi}{12})$.
b. Here $A = -3, \alpha = \frac{1}{2}$ and $\beta = 0$. So the amplitude is 3, the period is $\frac{2\pi}{\frac{1}{2}} = 4\pi$, the frequency is $\frac{1}{4\pi}$ cycles per unit time and there is no phase shift. As in a., I will draw the graph in stages, starting from $y = \cos x$, $0 \leq x \leq 2\pi$.



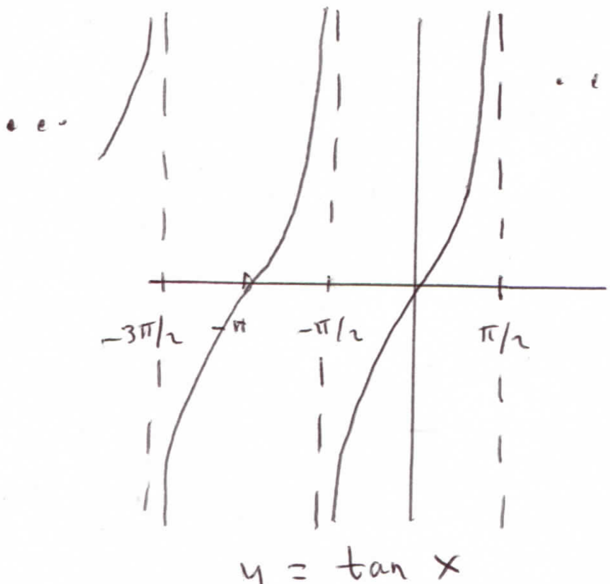


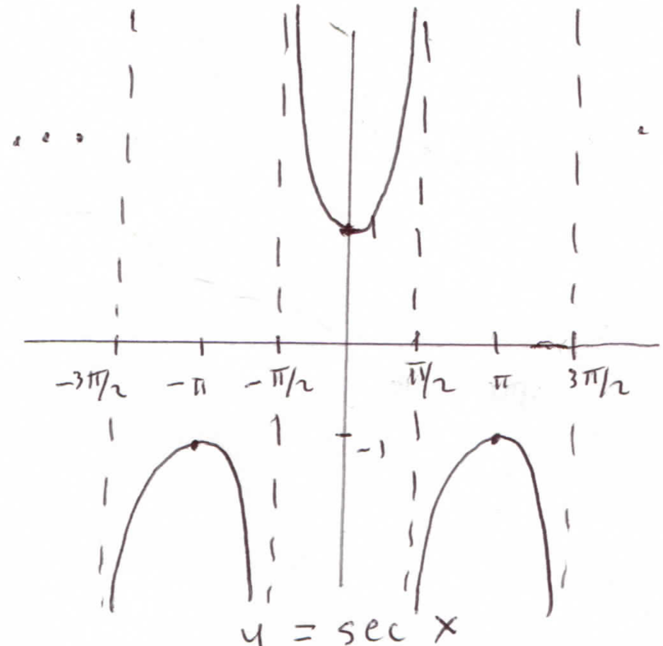




Stare at the graph for the remaining information requested: the maximum is 6, the minimum is 0, the graph is increasing on $(0, 2\pi)$, and decreasing on $(2\pi, 4\pi)$. ∎

5.36. Graphs of other trig functions.





You should be able to draw the graphs of csc as a translation of sec and cot as a translation and reflection of tan, using 3.3 and the fact that, by Corollary 5.19,

$$\cot x = -\tan(x - \frac{\pi}{2}) \quad \text{and} \quad \csc x = \sec(x - \frac{\pi}{2}).$$

Examples 5.37. Graph each of the following for one period.

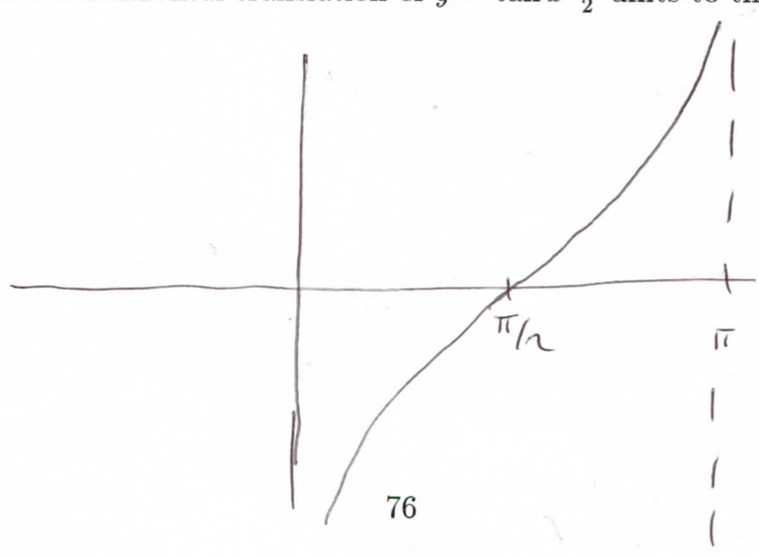
a. $f(x) = \tan(x + \frac{\pi}{2})$.

b. $f(x) = -\sec x$.

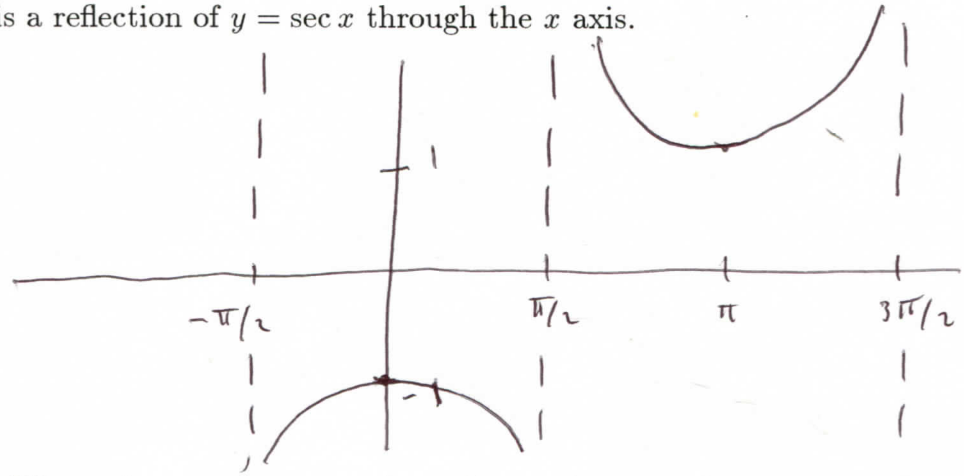
c. $f(x) = 1 + \csc x$.

Solutions. Refer to 3.3 for information about translations and reflections.

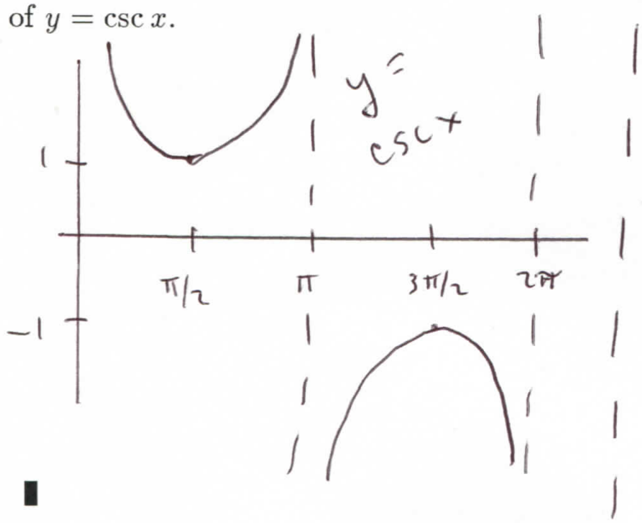
a. Here we have a horizontal translation of $y = \tan x$ $\frac{\pi}{2}$ units to the left.

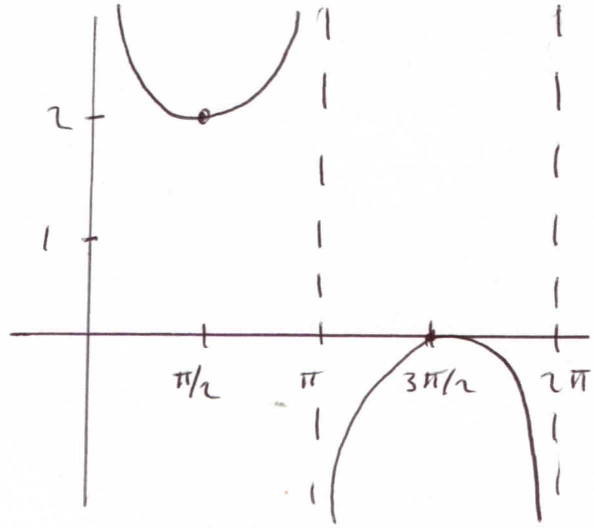


b. This is a reflection of $y = \sec x$ through the x axis.



c. First, I'll graph $y = \csc x = \sec(x - \frac{\pi}{2})$ as a horizontal translation, $\frac{\pi}{2}$ units to the right, of $y = \sec x$; then $y = 1 + \csc x$ will be a vertical translation, 1 unit up, of $y = \csc x$.



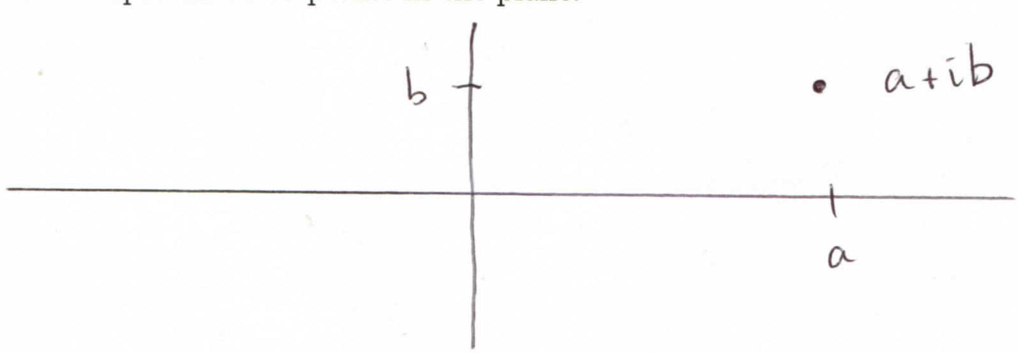


∎

5.38. Two-second introduction to complex numbers. i is a number (imaginary) such that $i^2 = -1$. The *complex numbers* are

$$\{a + ib \mid a, b \text{ are real }\}.$$

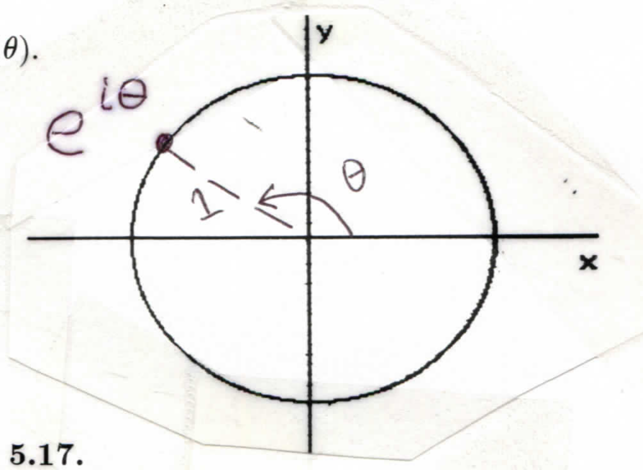
They can be represented as points in the plane.



a is called the *real* part of $a + ib$, and b is called the *imaginary* part.

de Moivre's formula is

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$



5.39. Derivation of Sum-of-angles formulas 5.17.

$$\begin{aligned}
&\cos(\alpha+\beta) + i\sin(\alpha+\beta) = e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}\\
&= (\cos(\alpha) + i\sin(\alpha))\,(\cos(\beta) + i\sin(\beta))\\
&= (\cos(\alpha))(\cos(\beta)) + (\cos(\alpha))(i\sin(\beta)) + (i\sin(\alpha))(\cos(\beta)) + (i\sin(\alpha))(i\sin(\beta))\\
&= ((\cos(\alpha))(\cos(\beta)) - (\sin(\alpha))(\sin(\beta))) + i\,((\cos(\alpha))(\sin(\beta)) + (\sin(\alpha))(\cos(\beta))),
\end{aligned}$$

so that, setting real parts equal to real parts, imaginary parts equal to imaginary parts, we have

$$\cos(\alpha+\beta) = (\cos(\alpha))(\cos(\beta)) - (\sin(\alpha))(\sin(\beta))$$

and

$$\sin(\alpha+\beta) = (\cos(\alpha))(\sin(\beta)) + (\sin(\alpha))(\cos(\beta)).$$

Also

$$\begin{aligned}
\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} &= \frac{[(\cos(\alpha))(\sin(\beta)) + (\sin(\alpha))(\cos(\beta))]}{[(\cos(\alpha))(\cos(\beta)) - (\sin(\alpha))(\sin(\beta))]}\\
&= \frac{[\tan(\beta) + \tan(\alpha)]}{[1 - \tan(\alpha)\tan(\beta)]},
\end{aligned}$$

where in the last step we divided the numerator and denominator by $\cos(\alpha)\cos(\beta)$.

5.40. Product-to-sum. For any real α, β,

(a)

$$(\sin\alpha)\,(\sin\beta) = \frac{1}{2}\,[\cos(\alpha-\beta) - \cos(\alpha+\beta)]\,,$$

(b)

$$(\sin\alpha)\,(\cos\beta) = \frac{1}{2}\,[\sin(\alpha-\beta) + \sin(\alpha+\beta)]\,,$$

and

(c)

$$(\cos\alpha)\,(\cos\beta) = \frac{1}{2}\,[\cos(\alpha-\beta) + \cos(\alpha+\beta)]\,.$$

Examples 5.41. Write each of the following as a sum, then simplify.
a. $\sin(\frac{5\pi}{12})\sin(\frac{\pi}{12})$.
b. $\cos(\frac{7\pi}{12})\sin(\frac{5\pi}{12})$.
c. $\cos(\frac{7\pi}{8})\cos(\frac{\pi}{8})$.
d. $\sin(\theta+\frac{\pi}{4})\sin(\theta-\frac{\pi}{4})$.

Solutions. a. Let $\alpha = \frac{5\pi}{12}$ and $\beta = \frac{\pi}{12}$, in 5.40(a):

$$\begin{aligned}\sin\left(\frac{5\pi}{12}\right)\sin\left(\frac{\pi}{12}\right) &= \frac{1}{2}\left[\cos\left(\frac{5\pi}{12}-\frac{\pi}{12}\right)-\cos\left(\frac{5\pi}{12}+\frac{\pi}{12}\right)\right]\\ &= \frac{1}{2}\left[\cos\left(\frac{\pi}{3}\right)-\cos\left(\frac{\pi}{2}\right)\right] = \frac{1}{2}\left[\frac{1}{2}-0\right] = \frac{1}{4}.\end{aligned}$$

b. Let $\alpha = \frac{5\pi}{12}$ and $\beta = \frac{7\pi}{12}$, in 5.40(b):

$$\begin{aligned}\sin\left(\frac{5\pi}{12}\right)\cos\left(\frac{7\pi}{12}\right) &= \frac{1}{2}\left[\sin\left(\frac{5\pi}{12}-\frac{7\pi}{12}\right)+\sin\left(\frac{5\pi}{12}+\frac{7\pi}{12}\right)\right]\\ &= \frac{1}{2}\left[\sin\left(-\frac{\pi}{6}\right)+\sin(\pi)\right] = \frac{1}{2}\left[-\frac{1}{2}+0\right] = -\frac{1}{4}.\end{aligned}$$

c. Now use 5.40(c):

$$\begin{aligned}\cos\left(\frac{7\pi}{8}\right)\cos\left(\frac{\pi}{8}\right) &= \frac{1}{2}\left[\cos\left(\frac{7\pi}{8}-\frac{\pi}{8}\right)+\cos\left(\frac{7\pi}{8}+\frac{\pi}{8}\right)\right]\\ &= \frac{1}{2}\left[\cos\left(\frac{3\pi}{4}\right)+\cos(\pi)\right] = \frac{1}{2}\left[-\frac{1}{\sqrt{2}}-1\right] = -\frac{(1+\sqrt{2})}{2\sqrt{2}}.\end{aligned}$$

d. Use 5.40(a) with $\alpha = (\theta+\frac{\pi}{4}), \beta = (\theta-\frac{\pi}{4})$:

$$\begin{aligned}\sin\left(\theta+\frac{\pi}{4}\right)\sin\left(\theta-\frac{\pi}{4}\right) &= \frac{1}{2}\left[\cos\left(\left(\theta+\frac{\pi}{4}\right)-\left(\theta-\frac{\pi}{4}\right)\right)-\cos\left(\left(\theta+\frac{\pi}{4}\right)+\left(\theta-\frac{\pi}{4}\right)\right)\right]\\ &= \frac{1}{2}\left[\cos\left(\frac{\pi}{2}\right)-\cos(2\theta)\right] = -\frac{1}{2}\cos(2\theta).\end{aligned}$$

∎

5.42. Sum-to-product. For any real α, β,

(a)

$$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right),$$

and

(b)

$$\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right).$$

Examples 5.43. Write each of the following as a product, and simplify.
a. $\sin(\frac{3\pi}{7}) - \sin(\frac{\pi}{14})$.
b. $\cos(\frac{3\theta}{2}) + \cos(\frac{\theta}{2})$.

Solutions. a. By 5.14(c), we are asked for

$$\sin\left(\frac{3\pi}{7}\right) + \sin\left(-\frac{\pi}{14}\right).$$

So use 5.42(a), with $\alpha = \frac{3\pi}{7}, \beta = -\frac{\pi}{14}$:

$$\begin{aligned}\sin\left(\frac{3\pi}{7}\right) + \sin\left(-\frac{\pi}{14}\right) &= 2\sin\left(\frac{1}{2}\left(\frac{3\pi}{7} + \left(-\frac{\pi}{14}\right)\right)\right)\cos\left(\frac{1}{2}\left(\frac{3\pi}{7} - \left(-\frac{\pi}{14}\right)\right)\right) \\ &= 2\sin\left(\frac{5\pi}{28}\right)\cos\left(\frac{\pi}{4}\right) = \sqrt{2}\sin\left(\frac{5\pi}{28}\right).\end{aligned}$$

b. Use 5.42(b), with $\alpha = \frac{3\theta}{2}, \beta = \frac{\theta}{2}$:

$$\begin{aligned}\cos\left(\frac{3\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right) &= 2\cos\left(\frac{1}{2}\left(\frac{3\theta}{2} + \frac{\theta}{2}\right)\right)\cos\left(\frac{1}{2}\left(\frac{3\theta}{2} - \frac{\theta}{2}\right)\right) \\ &= 2\cos(\theta)\cos(\frac{\theta}{2}).\end{aligned}$$

∎

5.44. Derivations of 5.20, 5.40, 5.42.
5.20: Let $\alpha \equiv \frac{\theta}{2}$.

$$\cos(\theta) = \cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha),$$

by a Sum-of-angles formula.

Now use the Pythagorean theorem to rewrite the last expression as either

$$(1 - \sin^2(\alpha)) - \sin^2(\alpha) = 1 - 2\sin^2(\alpha),$$

so that you can solve for $\sin\left(\frac{\theta}{2}\right) = \sin(\alpha)$, or

$$\cos^2(\alpha) - (1 - \cos^2(\alpha)) = 2\cos^2(\alpha) - 1,$$

so that you can solve for $\cos\left(\frac{\theta}{2}\right) = \cos(\alpha)$. ∎

For the tangent formula:

$$\begin{aligned}\frac{\sin(\theta)}{1+\cos(\theta)} &= \frac{\sin(2\alpha)}{1+\cos(2\alpha)} \\ &= \frac{2\sin(\alpha)\cos(\alpha)}{1 + (\cos^2(\alpha) - \sin^2(\alpha))} \\ &= \frac{2\sin(\alpha)\cos(\alpha)}{2\cos^2(\alpha)} \\ &= \frac{\sin(\alpha)}{\cos(\alpha)} = \tan(\alpha) = \tan(\frac{\theta}{2}).\end{aligned}$$

5.40:

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta),$$

while

$$\begin{aligned}\cos(\alpha - \beta) = \cos(\alpha + (-\beta)) &= \cos(\alpha)\cos(-\beta) - \sin(\alpha)\sin(-\beta) \\ &= \cos(\alpha)\cos(\beta) - \sin(\alpha)(-\sin(\beta)) \\ &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta).\end{aligned}$$

Adding those two expressions gives us the formula for $\cos(\alpha)\cos(\beta)$, while subtracting them gives us the formula for $\sin(\alpha)\sin(\beta)$. ∎

I'll leave it to you to do similar things with $\sin(\alpha + \beta)$ and $\sin(\alpha - \beta)$.

5.42: This follows from 5.40, with α replaced by $\frac{1}{2}(\alpha + \beta)$ and β replaced by $\frac{1}{2}(\alpha - \beta)$. ∎

When solving something like $\sin x = 0$, the solution is far from unique; x could be any integral multiple of π.

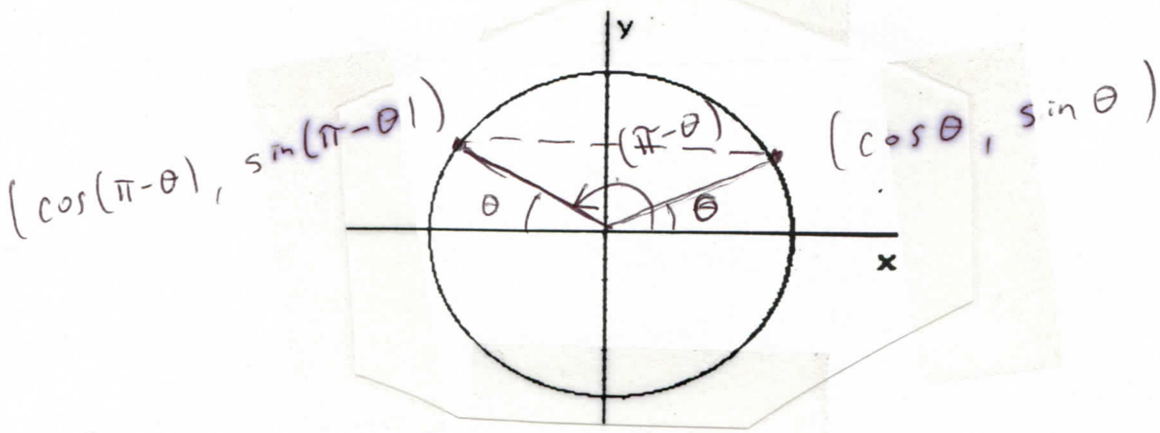
5.45. Trig equations. By a "trig equation" I mean one that can be rewritten as sine, cosine or tangent of an angle equal to a number. After getting one solution, you should be able to read off the others, by staring at the unit circle definition 5.10 and making an appropriate reflection, or by using the identities in 5.14. Here are general rules.

(a) If $x = \theta$ is one solution of

$$\sin x = y,$$

then

$$\{\text{ all solutions }\} = \{x \mid x = 2k\pi + \theta \text{ or } x = 2k\pi + (\pi - \theta), \text{ for some integer } k\}.$$



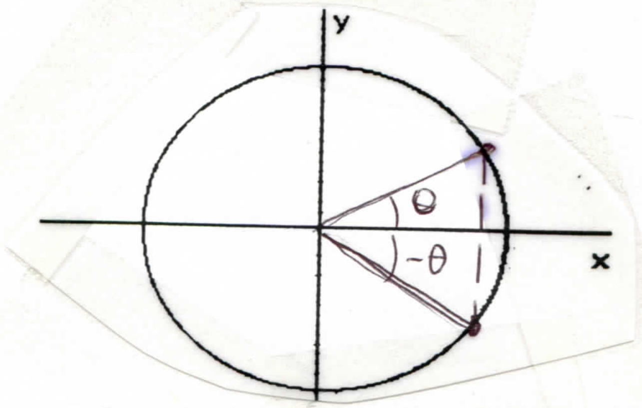
Note that we obtain $(\pi - \theta)$ from θ, on the unit circle, by reflecting through the y axis.

(b) If $x = \theta$ is one solution of

$$\cos x = y,$$

then

$$\{\text{ all solutions }\} = \{x \,|\, x = 2k\pi \pm \theta \text{ for some integer } k\}.$$



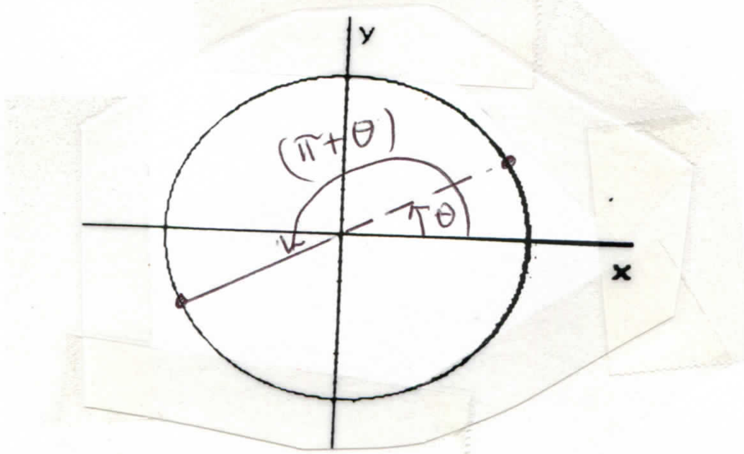
Note that, on the unit circle, $-\theta$ is the reflection of θ through the x axis.

(c) If $x = \theta$ is one solution of

$$\tan x = y,$$

then

$$\{\text{ all solutions }\} = \{x \,|x = k\pi + \theta, \text{ for some integer } k\}.$$



Examples 5.46. a. Find all solutions of $\sin x = -\frac{\sqrt{3}}{2}$.
b. Find all solutions of $\tan x = 1$.
c. Find all solutions of $2\cos(3x) = 1$ such that $-2\pi \le x \le \pi$.
d. Find all solutions of $2\sin^2\theta + \sin\theta = 1$.
e. Find all intercepts of the graph of $f(x) = 4\sin(2x + \frac{\pi}{3}) + 2$.

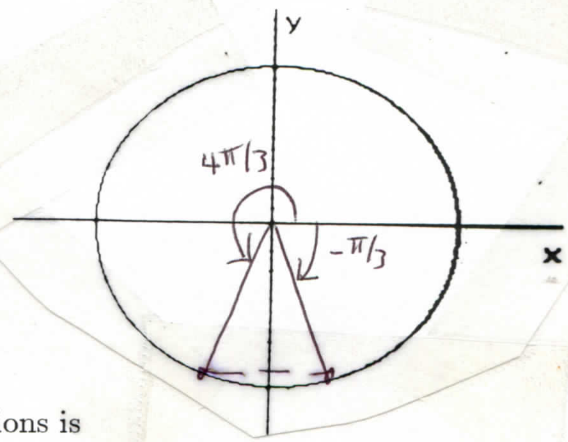
Solutions. a. You should have it in your memory that $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$. By 5.14(c), $\sin(-\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$.

Thus

$$x = -\frac{\pi}{3}$$

is one solution of $\sin x = -\frac{\sqrt{3}}{2}$. To get our other solutions, first reflect the angle $-\frac{\pi}{3}$ through the y axis to get another solution

$$x = (\pi - (-\frac{\pi}{3})) = \frac{4\pi}{3}.$$



Then the set of all solutions is

$$\{x \mid x = 2k\pi - \frac{\pi}{3} \quad \text{or} \quad x = 2k\pi + \frac{4\pi}{3}, \quad \text{for some integer } k\}.$$

(See 5.45(a).)

b. The famous angle $x = \frac{\pi}{4}$ satisfies $\tan x = 1$. So

$$x = \frac{\pi}{4}$$

is one solution of $\tan x = 1$; to get all solutions, we just add on integral multiplies of π:

$$\{x \mid x = k\pi + \frac{\pi}{4} \quad \text{for some integer } k\}.$$

(See 5.45(c).)

c. Let's isolate the cos, so we can use 5.45(b):

$$\cos(3x) = \frac{1}{2}. \qquad (*)$$

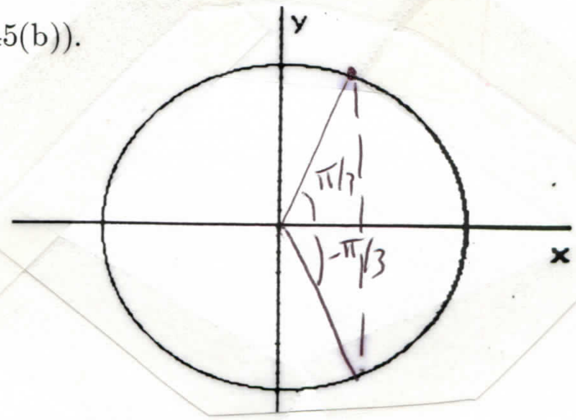
We'll put off worrying about $-2\pi \le x \le \pi$; first I'll write down all solutions of (*).
Since $\cos(\frac{\pi}{3}) = \frac{1}{2}$,

$$3x = \frac{\pi}{3}$$

gives one solution of (*). Reflecting through the x axis tells us that

$$3x = -\frac{\pi}{3}$$

will be another solution (see 5.45(b)).



Now we add on integral multiples of 2π; solutions of (*) look like

$$3x = 2k\pi \pm \frac{\pi}{3}$$

or

$$x = \frac{2k\pi}{3} \pm \frac{\pi}{9} \quad (k = \text{ integer }).$$

Now I'll worry about $-2\pi \le x \le \pi$. Start with $\frac{\pi}{9}$, then add on positive integral multiples of $\frac{2\pi}{3}$ until we get something bigger than π:

$$\frac{\pi}{9}, \frac{\pi}{9} + \frac{2\pi}{3} = \frac{7\pi}{9}, \frac{\pi}{9} + 2\left(\frac{2\pi}{3}\right) = \frac{13\pi}{9} \ldots$$

the last one is bigger than π, so we throw it out. Thus two acceptable solutions are $\frac{\pi}{9}$ and $\frac{7\pi}{9}$.

Similarly, let's start at $-\frac{\pi}{9}$ and subtract positive integral multiples of $\frac{2\pi}{3}$:

$$-\frac{\pi}{9}, -\frac{\pi}{9} - \frac{2\pi}{3} = -\frac{7\pi}{9}, -\frac{\pi}{9} - 2\left(\frac{2\pi}{3}\right) = -\frac{13\pi}{9}, -\frac{\pi}{9} - 3\left(\frac{2\pi}{3}\right) = -\frac{19\pi}{9} \ldots$$

The last one is smaller than -2π. So I throw it out. We've gotten three more acceptable solutions: $-\frac{\pi}{9}, -\frac{7\pi}{9}$ and $-\frac{13\pi}{9}$.

We should write down all our solutions in one place:

$$x = \pm\frac{\pi}{9}, \pm\frac{7\pi}{9}, \text{ or } -\frac{13\pi}{9}.$$

d. If we let $y \equiv \sin(\theta)$, we have a quadratic equation

$$2y^2 + y - 1 = 0$$

which we can solve by factoring:

$$0 = (2y-1)(y+1) \quad \rightarrow \quad y = \frac{1}{2} \text{ or } y = -1.$$

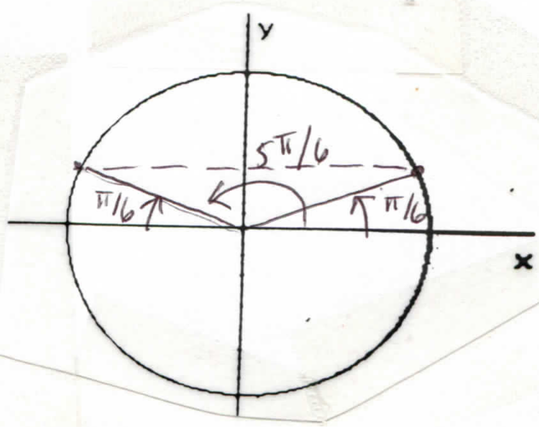
This means we have two trig equations to solve:

$$\sin(\theta) = \frac{1}{2} \tag{1}$$

and

$$\sin(\theta) = -1. \tag{2}$$

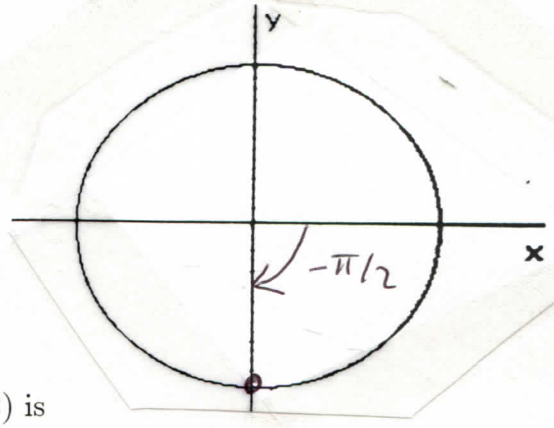
We solve both (1) and (2) with 5.45(a). For (1), we have one solution of $\theta = \frac{\pi}{6}$; reflecting through the y axis gives us another solution of $(\pi - \frac{\pi}{6}) = \frac{5\pi}{6}$.



The set of all solutions of (1) is

$$\{\theta \,|\, \theta = 2k\pi + \frac{\pi}{6} \quad \text{or} \quad \theta = 2k\pi + \frac{5\pi}{6}, \quad \text{for some integer } k\}.$$

For (2), we have one solution of $\theta = -\frac{\pi}{2}$; reflecting through the y axis gives us—the same angle.



Thus the set of all solutions of (2) is

$$\{\theta \mid \theta = 2k\pi - \frac{\pi}{2} \quad \text{for some integer } k\}.$$

A decent concern for the person grading a problem dictates that you conclude with your complete answer all in one place; the set of all solutions of d. is

$$\{\theta \mid \theta = 2k\pi + \frac{\pi}{6} \quad \text{or} \quad \theta = 2k\pi + \frac{5\pi}{6}, \quad \text{or} \quad \theta = 2k\pi - \frac{\pi}{2} \quad \text{for some integer } k\}.$$

e. We talked about intercepts back in 2.8. The y intercept is merely $f(0) = 4\sin(\frac{\pi}{3}) + 2 = 2(\sqrt{3}+1)$.

For the x intercepts, w set $f(x) = 0$ and solve for x:

$$f(x) = 0 \quad \rightarrow \quad \sin(2x + \frac{\pi}{3}) = -\frac{1}{2}.$$

We use 5.45(a). One solution of $\sin(2x + \frac{\pi}{3}) = -\frac{1}{2}$ is

$$\left(2x + \frac{\pi}{3}\right) = -\frac{\pi}{6}.$$

Reflecting through the y axis tells us that

$$\left(2x + \frac{\pi}{3}\right) = (\pi - (-\frac{\pi}{6})) = \frac{7\pi}{6}$$

is another solution. All solutions have the form

$$\left(2x + \frac{\pi}{3}\right) = 2k\pi - \frac{\pi}{6} \quad \text{or} \quad \left(2x + \frac{\pi}{3}\right) = 2k\pi + \frac{7\pi}{6},$$

for some integer k. Solve for x:

$$\left(2x + \frac{\pi}{3}\right) = 2k\pi - \frac{\pi}{6} \quad \rightarrow \quad 2x = 2k\pi - \frac{\pi}{2} \quad \rightarrow \quad x = k\pi - \frac{\pi}{4},$$

$$\left(2x + \frac{\pi}{3}\right) = 2k\pi + \frac{7\pi}{6} \quad \rightarrow \quad 2x = 2k\pi + \frac{5\pi}{6} \quad \rightarrow \quad x = k\pi + \frac{5\pi}{12}.$$

Recall that these are all x intercepts of the graph of f:

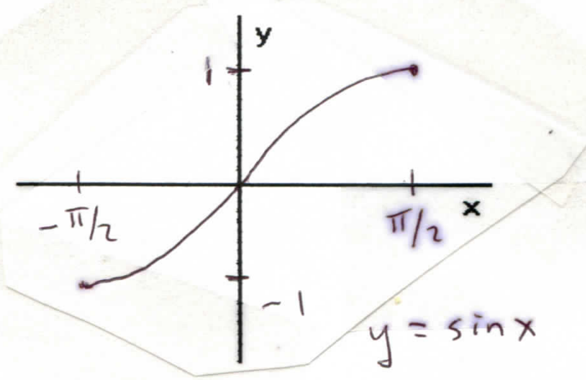
$$\{x \mid x = k\pi - \frac{\pi}{4} \quad \text{or} \quad x = k\pi + \frac{5\pi}{12}, \quad \text{for some integer } k\}.$$

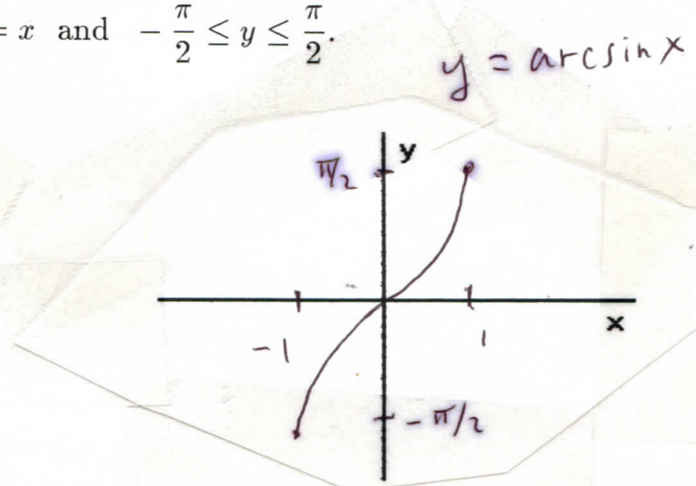
∎

As we have just seen, when solving trig equations, sine and cosine *really* fail to be one-to-one (see 2.42.) If we think of $f(x) = x^2$ as being two-to-one (two points get assigned to any nonzero point), then sine and cosine are ∞-to-one. To define inverse functions (see 2.38) of sine or cosine, we must severely restrict the domains.

5.47. arcsine, or $\sin^{-1}$, is the inverse function of $f(x) \equiv \sin x$, domain of $f \equiv [-\frac{\pi}{2}, \frac{\pi}{2}]$. In other words,

$$y = \arcsin x \quad \text{if} \quad \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$$





Examples 5.48. a. Find $\arcsin(-\frac{1}{2})$.
b. Find $\sin^{-1}(1)$.

Solutions. a. Let's write $y \equiv \arcsin(-\frac{1}{2})$ for what we want. Then, by definition of arcsin as the inverse function of a restriction of sin, we have

$$\sin y = -\frac{1}{2}.$$

This is a trig equation, which 5.45(a) tells us how to solve:

$$y = -\frac{\pi}{6} + 2k\pi \quad \text{or} \quad y = \frac{5\pi}{6} + 2k\pi, \quad \text{for some integer } k.$$

We choose the unique solution y that is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$:

$$y = \arcsin(-\frac{1}{2}) = -\frac{\pi}{6}.$$

b. Let $y \equiv \sin^{-1}(1)$. Then

$$\sin y = 1.$$

By 5.45(a),

$$y = 2k\pi + \frac{\pi}{2},$$

for some integer k. Since we want $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, we choose

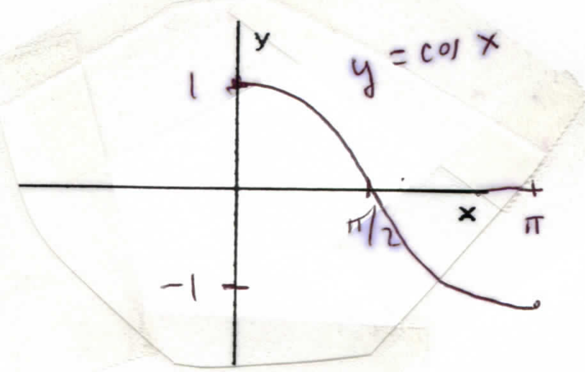
$$y \equiv \sin^{-1}(1) = \frac{\pi}{2}.$$

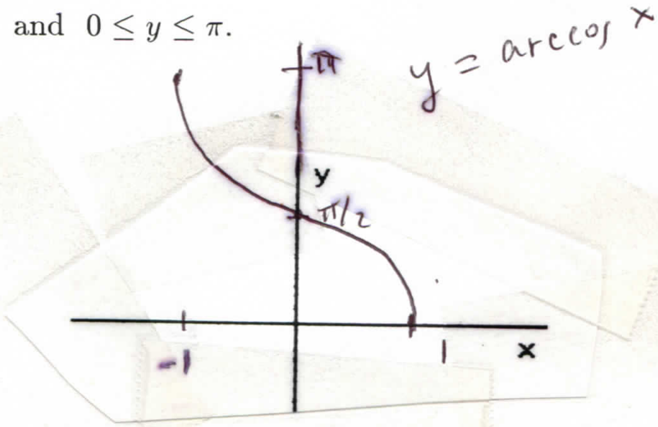
∎

We may similarly define inverse functions of (restrictions of) other trig functions.

5.49. arccosine, or $\cos^{-1}$, is the inverse function of $f(x) \equiv \cos x$, domain of $f \equiv [0, \pi]$. In other words,

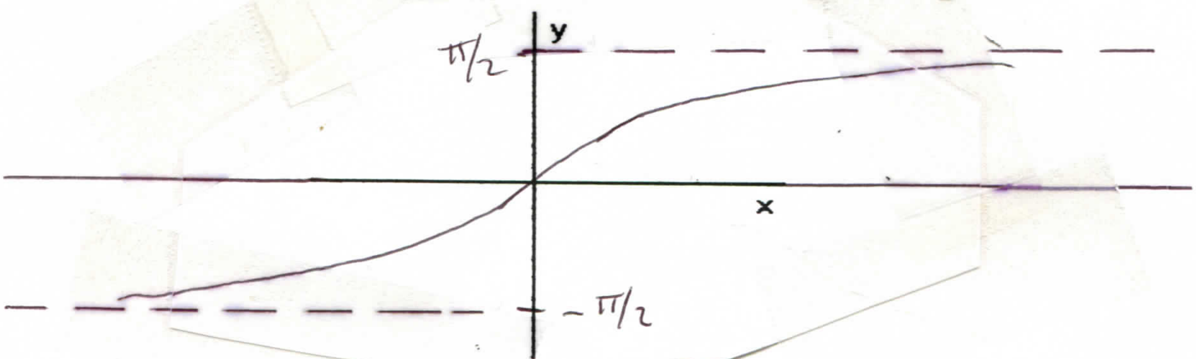
$$y = \arccos x \quad \text{if} \quad \cos y = x \;\text{ and }\; 0 \leq y \leq \pi.$$





5.50. arctangent, or $\tan^{-1}$, is the inverse function of $f(x) \equiv \tan x$, domain of $f \equiv (-\frac{\pi}{2}, \frac{\pi}{2})$. In other words,

$$y = \arctan x \quad \text{if} \quad \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}.$$



Examples 5.51. a. Find $\arccos(-\frac{1}{2})$.

b. Find $\arctan(\sqrt{3})$.

c. Find $\tan(\arccos(\frac{1}{3}))$.

d. For arbitrary x, find $\sin(\arctan x)$, and simplify.

Solutions. a. Let $y \equiv \arccos(-\frac{1}{2})$. Then

$$\cos y = -\frac{1}{2}.$$

By 5.45(b), we may solve this trig equation:

$$y = 2k\pi \pm \frac{2\pi}{3},$$

for some integer k. Our definition of arccos specifies that y must be between 0 and π. So we choose

$$\arccos(-\frac{1}{2}) \equiv y = \frac{2\pi}{3}.$$

b. Let $y \equiv \arctan(\sqrt{3})$. Then

$$\tan y = \sqrt{3}.$$

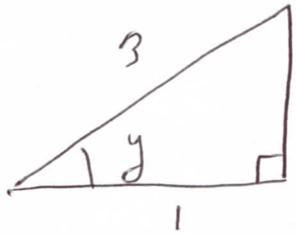
Checking our famous angles 5.13 gives us $y = \frac{\pi}{3}$ as one solution; since $\frac{\pi}{3}$ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, that's the solution we want:

$$\arctan(\sqrt{3}) = \frac{\pi}{3}.$$

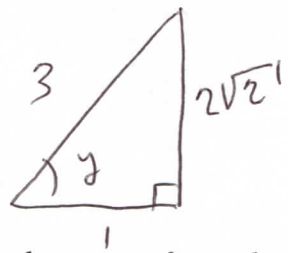
c. This takes some unravelling. Let $y \equiv \arccos(\frac{1}{3})$. Then

$$\cos y = \frac{1}{3}.$$

Draw a right triangle with such an angle (see 5.22):



We want $\tan y$, so let's fill in the triangle with the Pythagorean theorem; the missing side has length $\sqrt{3^2 - 1^2} = 2\sqrt{2}$.



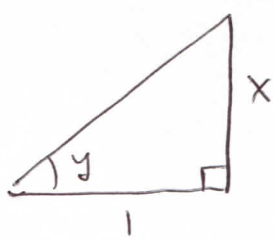
We can read off our desired tangent from that triangle:

$$\tan(\arccos(\frac{1}{3})) \equiv \tan y = \frac{2\sqrt{2}}{1} = 2\sqrt{2}.$$

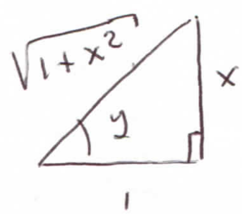
d. This is the same strategy as c. Let $y \equiv \arctan x$, so that

$$\tan y = x.$$

Here's what that looks like on a right triangle:



Again by the Pythagorean theorem, the hypotenuse of that triangle is $\sqrt{1 + x^2}$.



So we can read off our desired sine:

$$\sin(\arctan x) \equiv \sin y = \frac{x}{\sqrt{1 + x^2}}.$$

∎

CHAPTER I HOMEWORK

1. Find the distance between 2 and -7.

2. Solve $\frac{6}{7} = \frac{3}{x}$.

3. Solve $\frac{1}{2x} - 1 = \frac{1}{4}$.

4. Solve $3 - \frac{y}{2} = 2y - 4$.

5. Suppose a is a number whose absolute value is 7. What is the absolute value of $-a$?

6. Solve and graph:

a. $2x - 7 \geq -3x + 9$.

b. $|2x + 4| > 5$.

c. $3 + |x - 1| \leq 2$.

d. $-|1 - \frac{x}{4}| + 10 \geq 4$.

7. Find all x such that the distance from x to -5 is less than 3. Graph your solution.

8. For each of the following, either factor or assert that it is irreducible over the integers.

a. $(27z^3 - 8x^6)$.

b. $4x^2 - 12x + 9$.

c. $y^2 - 3y - 4$.

d. $2x^2 - x - 3$.

e. $x^2 + 4x - 3$.

9. Simplify: a. $(\sqrt{2})^6$ b. $\frac{(3^{2-\sqrt{2}})^4}{9}$ c. $(3^{2+\sqrt{5}})^2(9^{1-\sqrt{5}})$ d. $((\sqrt{2})^{-2})^{-1}$

e. $\frac{4+\sqrt{8}}{2}$ f. $\frac{a}{\sqrt{a}}$ g. $\frac{(a^5)^2}{a^6\sqrt{a}}$ h. $\frac{1}{\sqrt{a}}(a^2 - 3\sqrt{a})$.

10. Complete the square.

a. $x^2 + 9x - 1$.

b. $2y^2 - 20y + 9$.

11. Solve by completing the square.

a. $x^2 = 8x + 10$.

b. $3y^2 - 18y + 1 = 0$.

12. Solve by factoring: $x^2 - 7x = 18$.

13. Solve $2y^2 + 7y = 1$.

14. Without solving, determine how many real solutions the equation has.

a. $9z^2 + 12z = -4$.

b. $3x^2 + x + 1 = 0$.

15. Factor $x^3 - 7x + 6$.

16. Write $\frac{x^3+x^2+1}{x+1}$ as the sum of a polynomial and a fraction with the degree of the numerator less than the degree of the denominator.

17. Solve and graph.

a. $(z+1)^2(z-1)(z-5)(z^2+1) > 0$.

b. $x^2 + x \leq 6$.

c. $\frac{y(1-y^2)}{(y^2-4)} \leq 0$.

18. Solve

a. $4(3^x) = 36$.

b. $2(4^x) + 1 = 2$.

19. Find the distance between $(-5, 1)$ and $(2, 0)$ and the midpoint of the line between them.

20. If $(1, 2)$ is the midpoint of the line between $(-4, -1)$ and (a, b), what are a and b?

21. Suppose I start at noon 50 miles behind a bus going 60 miles per hour. Three hours later, I'm 10 miles behind the bus. How fast am I going? When will I catch the bus?

22. Find the equation of the circle centered at $(1, 4)$ with radius 2.

23. Find the center and radius of the circle satisfying the equation

$$2x^2 - 8x + 2y^2 + 6y - 1 = 0.$$

Graph the circle.

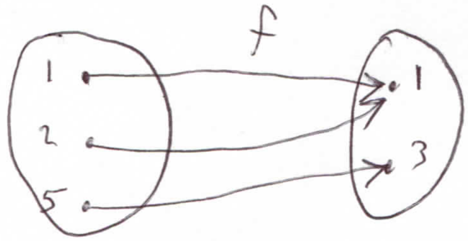
1. Suppose $f(x) = x^2, g(x) = 5x - 2$ and

$$h(x) = \begin{cases} -1 & -2 < x \leq 2 \\ 2^x & 2 < x \leq 4 \\ 1 - 2x & x \geq 6 \end{cases}$$

Find each of the following. For some, the answer might be "undefined."

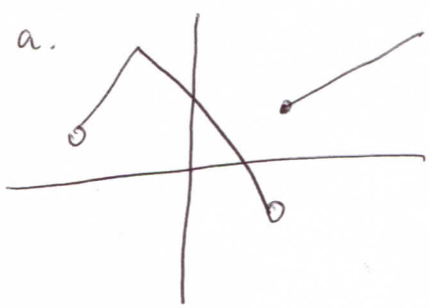
a. $f(-3)$. b. $g(2)$. c. $f(g(2))$. d. $g(f(2))$. e. $h(-3)$. f. $f(z)$. g. $f(y^3)$. h. $h(f(1))$. i. $h(g(0))$. j. $g(h(0))$. k. $g(f(x))$. ℓ. $f(g(x))$.

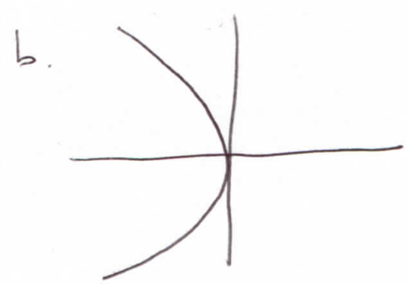
2. For the following function f represented by a "blob diagram," represent the function as a table, state the graph (as a set of ordered pairs) and draw the graph (in the Cartesian plane). Find the domain and range.



3. Suppose a function f has the graph $\{(-1, 2), (0, 1), (1, 1), (3, 0)\}$. Draw the graph, and represent the function as a table and as a "blob diagram"(as in number 2). Find the domain and range. What is $f(0)$?

4. Which of the following are graphs of a function?





5. Find the intercepts of the graphs of each of the following functions (do not graph the functions).

a. $f(x) = x^2 + x + 5$.

b. $g(x) = 2x^2 - x - 2$.

c. $h(x) = \frac{2x+1}{x-1}$.

6. Find the domain of each of the following functions.

a. $f(x) = \frac{1}{\sqrt{(x-3)^2(x+1)(x-1)}}$.

b.

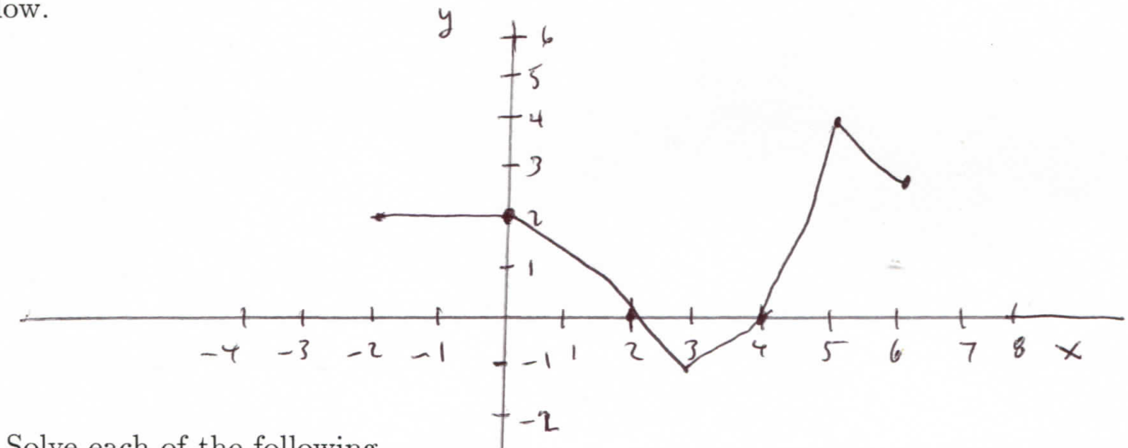
$$g(x) = \begin{cases} \frac{1}{x-2} & -3 \leq x \leq 5 \\ \sqrt{x} & 6 < x < 8 \\ 2 & x \geq 8 \end{cases}$$

7. Find the range of each of the following functions.

a. $f(x) = 2x^2 - x$.

b. $g(x) = \frac{1-x}{3x+1}$.

8. Estimate the domain, range and intercepts of the function f from its graph below.



9. Solve each of the following.

a. $1 - 3|2x+1| > -11$.

b. $2 - \frac{1}{4}(3^{\frac{2z}{5}}) = -\frac{19}{4}$.

10. Find the function whose graph is a straight line that:

a. goes through $(-1, -3)$ and has slope 19;

b. goes through $(1, 0)$ and $(0, 1)$;

c. goes through $(-1, 2)$ and is parallel to $2x - 4y = 21$;

d. goes through $(-1, 2)$ and is perpendicular to $2x - 4y = 21$;

e. goes through $(1, 2)$ and is parallel to $y = 5$;

11. For each of the following, graph $y = f(x)$ and find the vertex and any intercepts of the graph.

a. $f(x) = 12x - 2x^2 + 5$.

b. $f(x) = 4x^2 + 4x - 1$.

c. $f(x) = x^2 - \frac{1}{2}x$.

12. Find the linear function f such that $f(1) = 2$ and $f(-1) = 3$.

13. Find the linear function g such that $g(-2) = 5$ and the slope of the graph of g equals 3.

14. There are two popular ways to measure temperature, Celsius and Fahrenheit. Degrees Celsius is a linear function of degrees Fahrenheit, 0 degrees Celsius equals 32 degrees Fahrenheit, and 100 degrees Celsius equals 212 degrees Fahrenheit. Write degrees Celsius as a function of degrees Fahrenheit.

15. Suppose that body fat is a linear function of the amount of candy you eat each day. If one gram of candy per day produces 10 pounds of body fat and 3 grams of candy per day produces 40 pounds of body fat, how much body fat will 8 grams of candy per day produce?

16. Suppose that plant growth is a linear function of water, 2 liters of water produces 10 centimeters cubed of plant, and 5 liters of water produces 20 centimeters cubed of plant.

a. How much plant will 4 liters of water produce?

b. If 12 centimeters cubed of plant is produced, how much water was used?

17. For each of the following, find the maximum or minimum and determine on which intervals the graph is decreasing.

a. $f(x) = 12x - 2x^2 + 5$.

b. $f(x) = 4x^2 + 4x - 1$.

c. $f(x) = x^2 - \frac{1}{2}x$.

18 . Find two numbers, whose product is a maximum, such that twice the first number added to the second number is twenty.

19 .When I fire a cannonball off the top of a building, then for any positive t, the cannonball is $100 + 160t - 16t^2$ feet above the ground after t seconds, until it hits the ground.

(a) When will the cannonball hit the ground?

(b) At what time will the cannonball attain its maximum height?

(c) What is the maximum height it achieves?

NOTE: this can be done without calculus.

20 .Suppose you have 200 feet of fencing. One side of a rectangular pasture will be your barn, the other three sides will use your fencing. What should the proportions

of your pasture be, to maximize the area of the pasture? (NOTE: no calculus is needed)

21. For each of the following functions f, find the average rate of change on the indicated interval and simplify.

a. $f(x) = 3x^2 - 2x + 5$ on $[a, b]$.

b. $f(x) = \frac{1}{x}$ on $[.5, 2]$.

c. $f(x) = \frac{1}{x}$ on $[a, b]$.

d. $f(x) = x^3$ on $[0, 1]$; on $[0, 2]$.

e. $f(x) = \frac{1}{\sqrt{x}}$ on $[a, b]$.

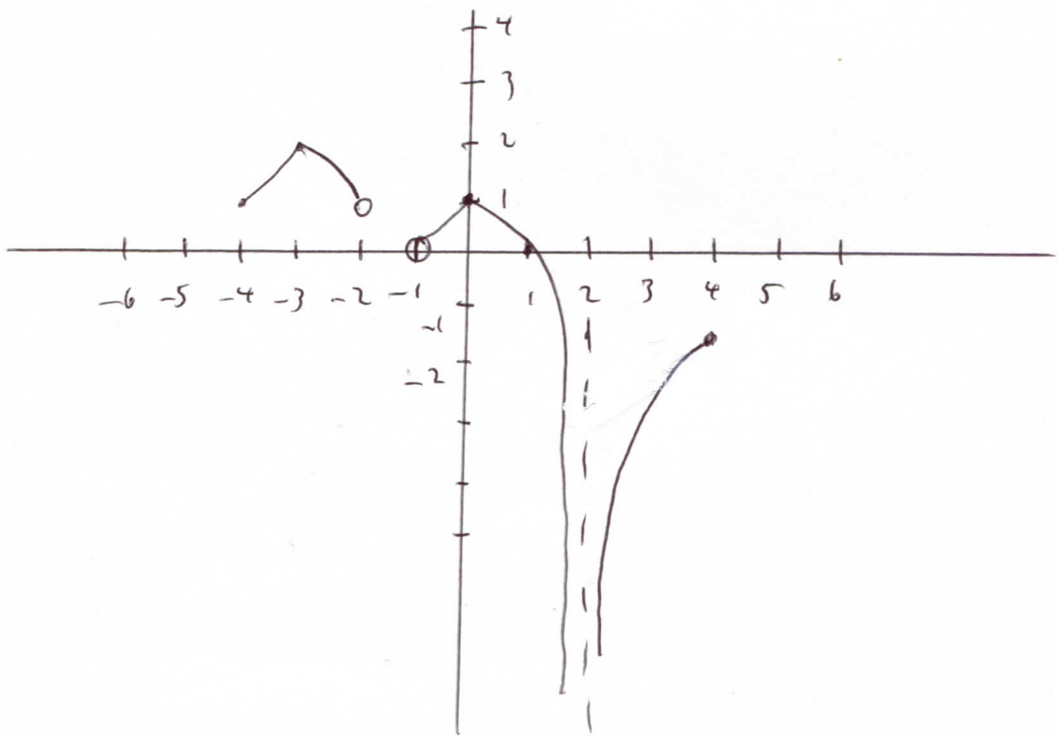
f. $f(x) = x^2$ on $[a, a + h]$.

22. Ten seconds after leaving the earth, a missile fired straight up is 60 yards above the ground. What is its average velocity over those ten seconds?

23. Use the following table to get the average rate of change of f on $[1, 4]$.

x	0	1	2	3	4	5
$f(x)$	9	-2	77	$\sqrt{2}$	3	100

24. Use the graph of $y = f(x)$ below to estimate the domain and range of f, intercepts of f, any maxima and minima of f, where they occur, and on what intervals f is increasing or decreasing.



25. If $g(x) = 1 - x^2$ and $f(x) = \sqrt{x}$, find $(f \circ g)$ and $(g \circ f)$, including domain.

26. If $f(x) = 2x^2$ and $g(x) = x + 2$, find a. $(f \circ g)(-1)$. b. $(g \circ f)(-1)$. c. $(g \circ f)(1)$. d. $(g \circ f)(x)$. e. $(f \circ g)(x)$.

27. Write $h(x) = \frac{1}{\sqrt{37x^3+19x}}$ as a composition of two simpler functions.

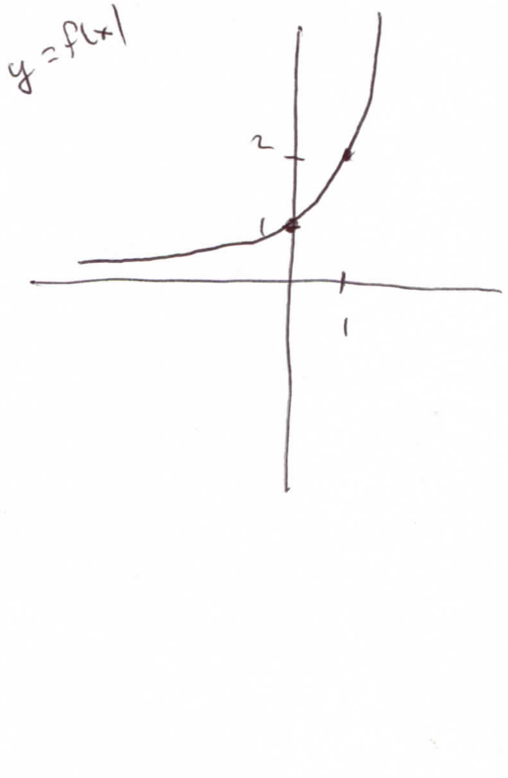
28. Find the graph of $(f \circ g)$ and $(g \circ f)$ if the graph of f is $\{(-1,0),(0,2),(1,0),(2,2)\}$ and the graph of g is $\{(-1,3),(0,1),(1,0)\}$.

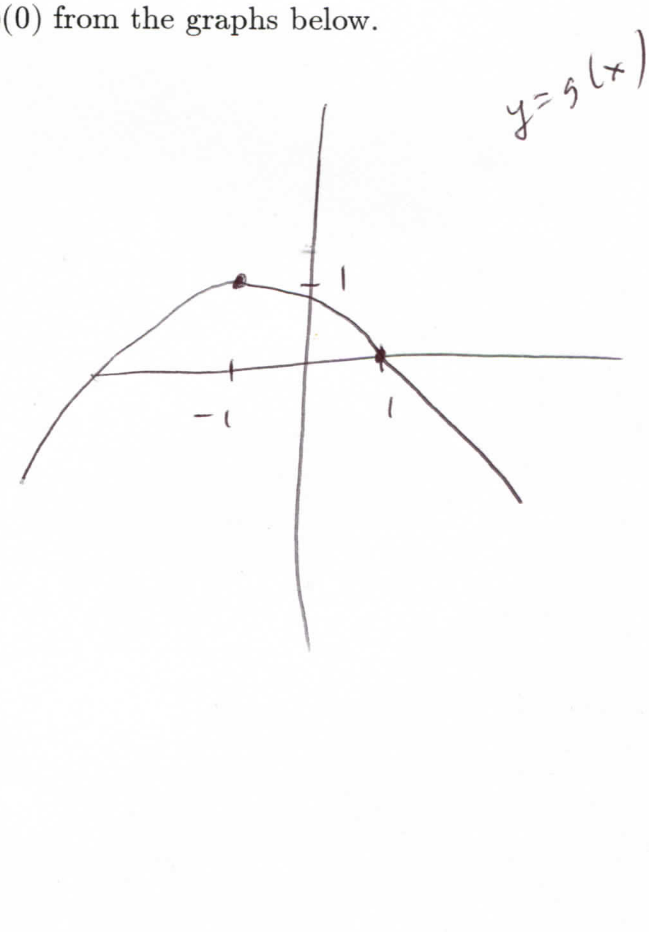
29. Find $(f \circ g)$ and $(g \circ f)$ if f and g are given by the following tables.

x	-1	0	1	2
$f(x)$	0	3	0	1

x	0	1	2
$g(x)$	0	3	3

30. Estimate $(f \circ g)(-1)$, $f^{-1}(2)$ and $(g \circ f)(0)$ from the graphs below.





31. Find f^{-1}, if f is given by the following table.

x	-1	0	1	3	5
$f(x)$	0	2	-1	5	4

32. Find the graph of f^{-1}, if the graph of f is $\{(-2,3),(-1,1),(0,2),(2,-1)\}$.

33. If $f(1) = -3$ and f has an inverse function, what is $f^{-1}(-3)$?

34. Find $f^{-1}(-1)$, if $f(x) = -x^2 + 4x + 9$, with domain of f equal to $(-\infty, 0)$.

35. Find $f^{-1}(2)$, if $f(x) = \frac{x+1}{2x-5}$.

36. In each of the following, find f^{-1}.

a. $f(x) = -2x + 5$.

b. $f(x) = x^2 + 2x + 9$, domain of f equal to $[0, \infty)$.

c. $f(x) = \frac{4-3x}{2x+1}$.

37. Determine, without calculating an inverse, which of the following functions have an inverse function.

a. $f(x) = \sqrt{1-x^2}$.

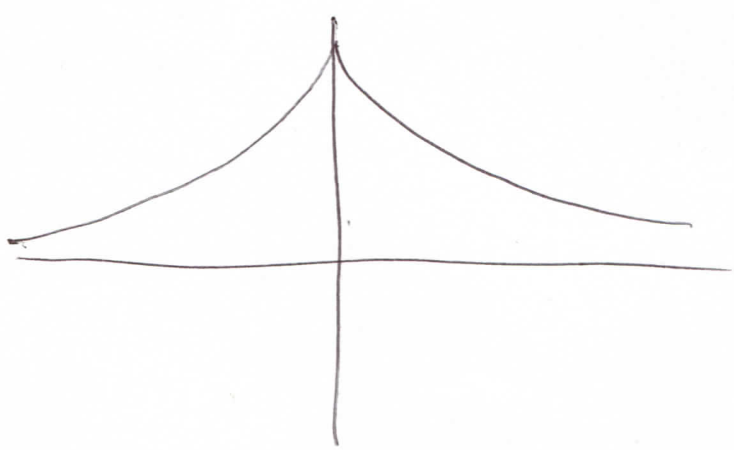
b. $f(x) = x^2 + 1$.

c. $f(x) = (x-1)^3 + 4$.

d. $f(x) = x^2 - 4x + 1$, domain of f equal to $[3, \infty)$.

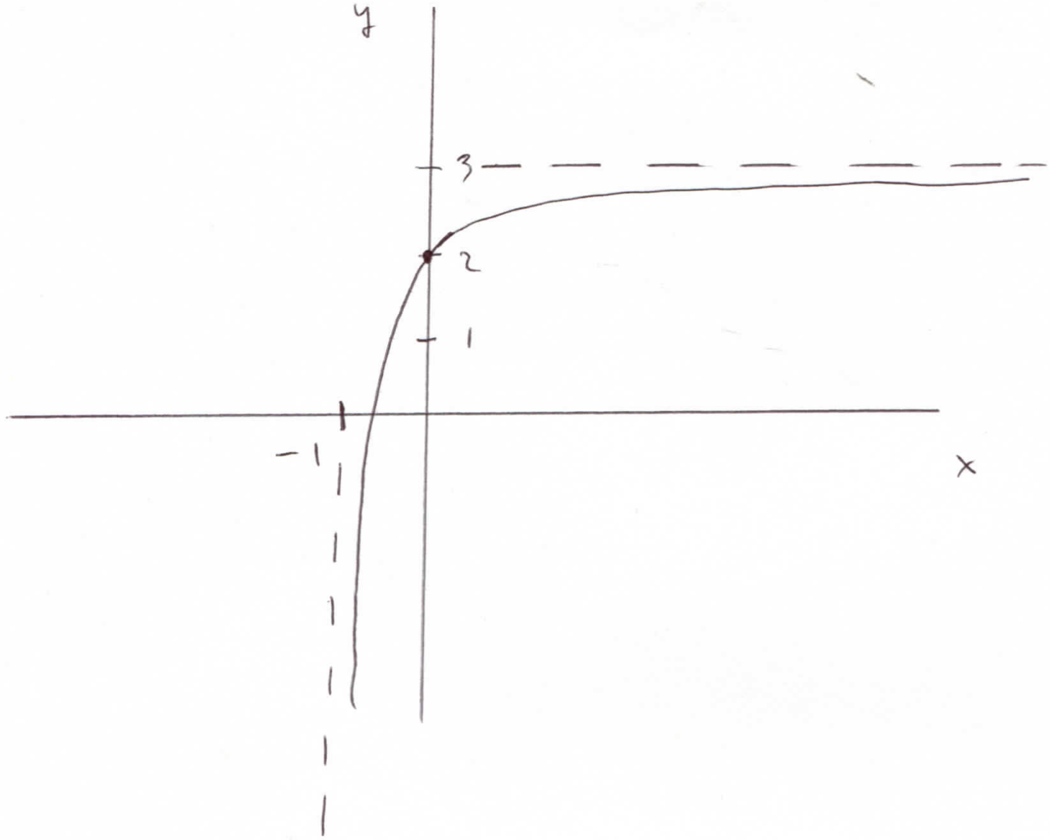
e. $f(x) = x^2 - 4x + 1$, domain of f equal to $[1, \infty)$.

f. f has the following graph.



38. Find the linear function f such that $f^{-1}(1) = 2, f^{-1}(0) = -1$.

39. Draw the graph of f^{-1}, given that the following is the graph of f.



CHAPTER III HOMEWORK

1. Graph each of the following.

a. $y = 2 - (x-1)^3$.

b. $y = \sqrt{2-x}$.

c. $y = |x+1| - 2$.

d. $y = 6x - 2x^2 + 2$.

e. $y = 2x - 4$.

f.

$$y = \begin{cases} |x| & x < -1 \\ x^3 & -1 \le x < 1 \\ \sqrt{x} & 4 < x \le 9 \end{cases}$$

g. $y = 3 + \frac{1}{x+2}$.

2. Find all intercepts in no. 1.

3. Find any symmetry of the graphs of the following equations, without drawing the graphs.

a. $x^2y - y^2x = 1$.

b. $x^3y + y^2x^2 = 1$.

4. Graph each of the following. For any parabola, identify the vertex. For any hyperbola, draw (dotted line) and identify any asymptotes.

a. $x - 2y^2 + 12y = 23$.

b. $x + y^2 + 2y = 0$.

c. $9x^2 + 18x + 4y^2 - 4y + 7 = 0$.

d. $4y^2 + 16y - x^2 + 16x = 64$.

e. $x^2 - 9y^2 + 36y = 45$.

5. Suppose your annual salary twenty years from now is a linear function of how many hours per week you spend on homework this quarter. Suppose further that five hours per week spent on homework this quarter leads to an annual salary twenty years from now of 30,000 dollars and eighteen hours per week spent on homework this quarter leads to an annual salary twenty years from now of 55,000 dollars.

a. If you spend thirty hours per week on homework this quarter, what will your annual salary be twenty years from now?

b. How many hours per week must you spend on homework this quarter, if you want your annual salary twenty years from now to be 40,000 dollars?

6. Find the domain, range and intercepts of the function represented by the following table. Identify its graph (as a set of ordered pairs) and draw its graph.

x	-1	0	2	3
$f(x)$	5	1	0	3

7. Solve and graph

$$\frac{x^2-1}{x^2-x-6} \geq 0.$$

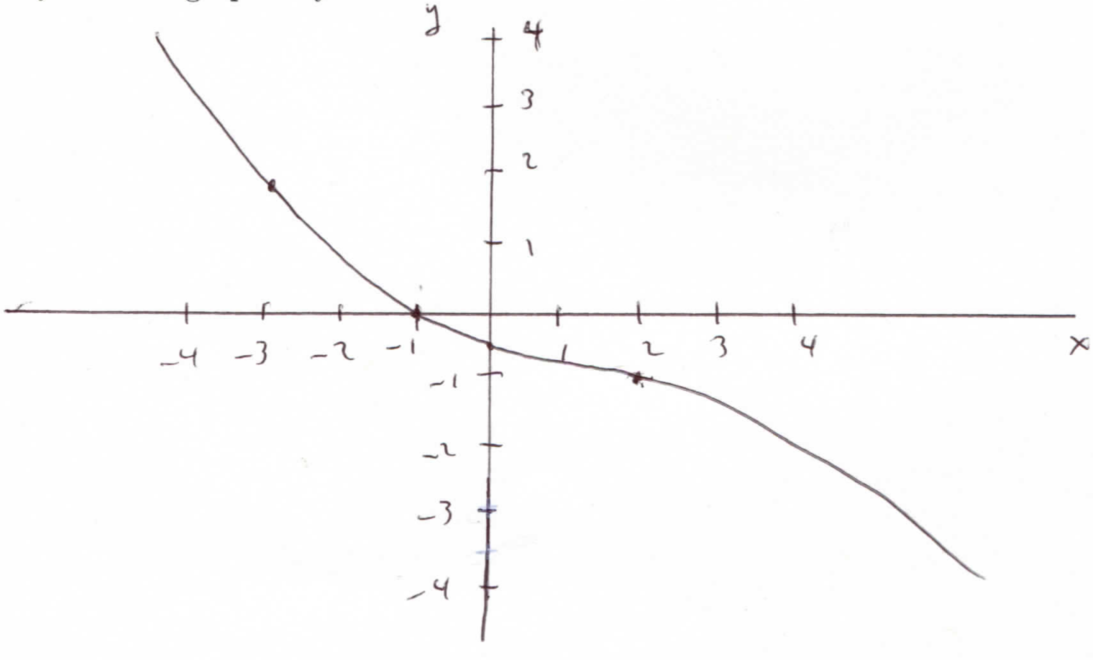
8. Suppose f has an inverse function and $f^{-1}(7) = 19$. What is $f(19)$?

9. What is the graph of f^{-1}, if the graph of f is $\{(-1,3),(0,1),(1,2)\}$?

10. If $f(y) = y^2$ and $g(z) = \frac{1}{z+1}$, find, and simplify if possible:

a. $f(z)$ b. $g(\theta)$ c. $(f \circ g)(y^2)$ d. $(g \circ f)(-1)$ e. $g(\frac{1}{x})$ f. $g(g(r))$.

11. Use the graph of f below to estimate $f^{-1}(2), f(2)$, and all intercepts of the graph of f and the graph of f^{-1}.



CHAPTER IV HOMEWORK

1. Suppose there are 50 bacteria at midnight, January 1, 1996, and the number of bacteria doubles every half hour. How many bacteria will there be at noon, January 1, 1996?

2. Suppose a radioactive isotope has a half-life of 2 seconds. If there is 100 grams of the isotope now, how much will there be in an hour?

3. Graph each of the following.

a. $y = 2^{-x}$.

b. $y = 5(3^x)$.

c. $y = 1 - (\frac{1}{3})^x$.

d. $y = 5 + 4^{x-2}$.

4. Suppose a population triples every fifty days. If we have 200 now, how many will there be after a year?

5. Suppose a radioactive isotope has a half-life of eight years. If there's currently five pounds, how much was there ninety years ago?

6. In each of the following, solve for the variable.

a. $\left(\frac{1}{4}\right)^x = 8$.

b. $2 \cdot 3^{2t-1} - 4 = 14$.

7. Write $h(x) = 2^{x^2}$ as a composition of two simpler functions.

8. Suppose $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = (1 - x^2)$. Find, and simplify if possible, $(f \circ g)(x)$ and $(g \circ f)(x)$.

9. Find f^{-1}, for the function f below.

x	-1	0	2	3
$f(x)$	5	0	1	2

10. In each of the following, find $f^{-1}(2)$.

a. $f(x) = 3x - 7$.

b. $f(x) = \frac{2x-3}{x+1}$.

c. $f(x) = 2x^2 - x + 1$, domain of f equal to $(-\infty, 0)$.

11. Suppose f has an inverse function and $f(-3) = 12$. What is $f^{-1}(12)$?

12. Solve for whatever unknown is hanging around. Simplify as much as possible.

a. $9\sqrt{3} = 3^x$.

b. $\frac{\sqrt{2}}{8} = 4^z$.

c. $5 = 3 + (\frac{1}{3})(2^y)$;

d. $\log_2(1 + x^2) = 5$

e. $9 = 2 + 7(3^{\frac{t}{4}})$

f. $9\log_3(5w - 1) - 5 = 13$

g. $\ln(2x^2) - \ln(x^2 + 1) = 0$.

h. $\ln(2x^2) - 2\ln(x + 1) = 0$.

i. $e^{2t-1} = 2$.

j. $2\log_4(x) - \log_4(x + 1) = 1$.

k. $5e^{2x} - 9 = 21$.

13. Simplify each of the following.

a. $\log_3(81\sqrt{3})$.

b. $\log_{\frac{1}{5}}(25)$.

c. $\frac{2^3\sqrt{2}}{4\sqrt{32}}$.

d. $\ln(\sqrt{e})$.

14. Suppose a population doubles every fifty days. If we have 200 now, how long will it take to grow to a million?

15. Suppose a radioactive isotope has a half-life of eight years. If there's currently five pounds, how long will it take to shrink to one pound? Simplify your answer.

16. Suppose a radioactive isotope has a half-life of eight years. If there's currently two pounds, how long ago was there fifty pounds? Simplify your answer.

17. Carbon-14 has a half-life of 5570 years. When an organism dies, the Carbon-14 starts decaying.

If a fossil contains only one percent of the Carbon-14 that it contained when it died, how old is the fossil? Simplify your answer.

18. If $f(x) = 2^x$ and $g(x) = x - \log_2 x$, find and simplify:

a. $(f \circ g)(z)$ b. $(g \circ f)(t)$ c. $f(x - g(x))$ d. $f(g(1) + 2)$ e. $g(f(2) - 2)$.

19. Solve $3 + 2|x + 5| \leq 7$.

20. Solve

$$\frac{(x+1)^2(x-2)}{(x-5)} \geq 0.$$

21. Complete the square: $y = \frac{1}{4}x^2 + 3x - 2$.

22. For each of the following functions, find the average rate of change over the indicated interval, and simplify.

a. $f(x) = 5(2^x)$, on $[0, h]$.

b. $f(x) = 5(2^x)$, on $[a, a + h]$.

c. $f(x) = 5x^2$, on $[a, a + h]$.

d. $f(x) = \log_2(x)$, on $[1, 8]$.

23. Graph each of the following.

a. $y = \ln(x + 1)$.

b. $y = e^{-|x|}$.

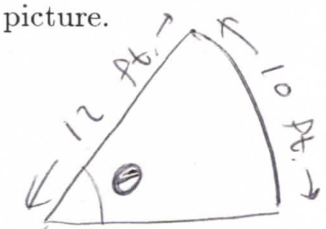
c. $y = 3 - \log_2(x - 1)$.

CHAPTER V HOMEWORK

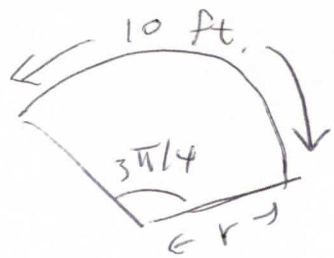
1. How many radians is 290 degrees?

2. How many degrees is $\frac{7\pi}{4}$ radians?

3. Find θ in the following picture.



4. Find the radius r in the following picture.



5. If you drive in a car at 50 miles per hour, and each of your tires has a radius of 2 feet, at what speed, in radians per minute, are your tires revolving?

6. If you spin a nunchaku at 200 revolutions per minute, with a radius of one foot, at what (linear) speed is the end of the nunchaku moving?

7. Suppose two wheels, one of radius 10 inches, one of radius 2 inches, have a belt running around both of them. If the larger wheel is revolving at 30 revolutions per minute, how fast, in revolutions per minute, is the smaller wheel revolving?

8. Find the area of a sector of angle $\frac{\pi}{6}$ radians, of a disc of radius 3 feet.

9. Suppose $\sin(\theta) = \frac{2}{7}$ and $-\frac{3\pi}{2} < \theta < -\frac{\pi}{2}$. What is $\cos(\theta)$?

10. Find the sine and cosine of each of the following angles (in radians):

$$-\frac{\pi}{2}, \frac{77\pi}{2}, 91\pi, -\frac{77\pi}{2}, -4\pi, \frac{19\pi}{2}.$$

11. Which of the following functions has an inverse function?

x	0	1	2	3	4
$f(x)$	19	7	$-\frac{1}{2}$	1	2

x	0	1	2	3	4
$g(x)$	1	2	3	1	2

12. Solve and graph: $x^2 - 2x > 8$.

13. Find each of the following.

a. $\sin(-\frac{19\pi}{4})$.
b. $\cos(\frac{25\pi}{3})$.
c. $\tan(-\frac{9\pi}{4})$.
d. $\cos(\theta)$, if $\tan(\theta) = -2$ and $\sin(\theta)$ is positive.
e. $\sin^2(\theta)$, if $\cot(\theta) = 5$.
f. $\csc(-\frac{\pi}{6})$.
g. $\tan(\frac{\pi}{12})$.
h. $\tan(\alpha - \beta)$, if $\tan(\alpha) = 3, \tan(\beta) = -2$.
i. $\sin(\alpha + \beta)$, if $\sin(\alpha) = \frac{1}{4}, \cos(\beta) = \frac{2}{3}, -\frac{\pi}{2} < \beta < 0$ and $\frac{\pi}{2} < \alpha < \pi$.
j. $\cos(\frac{11\pi}{12})$.
k. $\sin(19\pi - \theta)$, if $\sin(\theta) = \frac{1}{5}$.
ℓ. $\sin(\frac{\pi}{6} - \theta)$, if $\sin(\theta) = \frac{1}{5}$ and $\cos(\theta)$ is positive.
m. $\tan(\frac{\pi}{12})$.
n. $\cos(\frac{5\pi}{8})$.

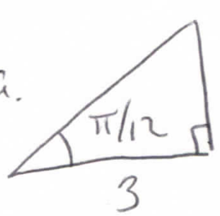
14. If $\tan(\alpha) = -3$ and $\frac{3\pi}{2} < \alpha < 2\pi$, find sine, cosine and tangent of $\frac{\alpha}{2}, \alpha$ and 2α.

15. Discuss the symmetries of the graphs $y = \sin x, y = \cos x$ and $y = \tan x$.

16. Graph $y = f(x) \equiv -\frac{1}{3}x^2 + x - 1$. Find the vertex and all intercepts. Find the maximum or minimum value of $f(x)$. Find on which intervals the graph of f is decreasing and where it is increasing. Find the range of f.

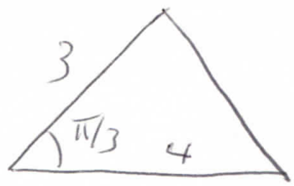
17. Find the lengths of all sides in each of the following triangles.

a.



b.

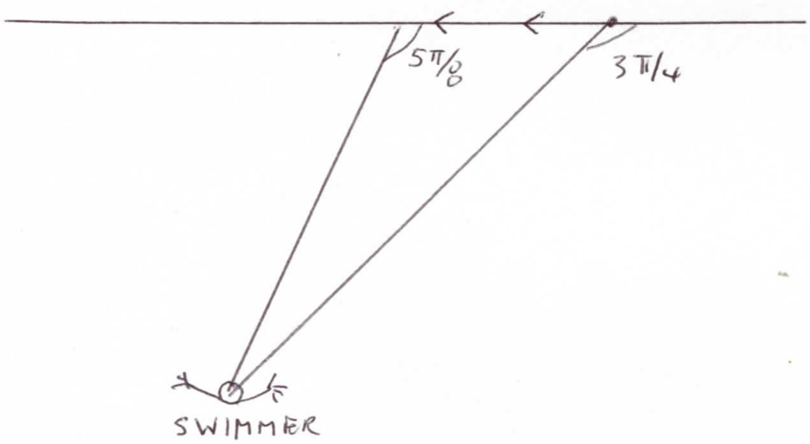
18. Find the area of the following triangle.



19. How long must a ladder be, if you want to lean it at a $\frac{5\pi}{12}$ angle and reach something 20 feet high?

20. When you walk two feet away from directly below a hovering pigeon, the angle of the pigeon from you is $\frac{\pi}{3}$. How high above your head is the pigeon?

21. Suppose you are on the straight shore of a lake and you see a swimmer drowning. Because you cannot swim far, you need to know how far away the swimmer is before you try to rescue him/her. You measure an angle of $\frac{3\pi}{4}$ between the shore and a line from you to the swimmer; then run a half mile down the beach and measure an angle of $\frac{5\pi}{8}$ between the shore and a line from you to the swimmer.



a. How far from your new position is the swimmer?

b. How far is the swimmer from the shore? In other words, what is the length of a perpendicular line to the shore from the swimmer?

22. Two roads begin at the same point and are at an angle of $\frac{\pi}{3}$ to each other. You've gone along one road 500 feet, and another person has gone 300 feet along the other road.

a. How far away are you from the other person?

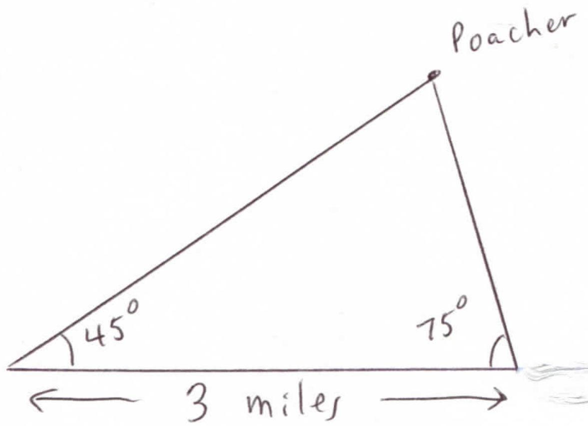
b. How far are you from the other road? In other words, what is the shortest distance from where you are to the other road?

23. Suppose you fire two rockets at the same time from the same point, one straight up at 100 feet per second, the other at an angle of $\frac{3\pi}{8}$ from the ground at 120 feet per second.

a. How far apart are the rockets after two minutes?

b. How high above the ground is the second rocket after two minutes?

24. Two observation towers, one to the west of the other, are three miles apart. Both towers see a poacher. The line from the western tower to the poacher makes an angle of 45 degrees with the line between the two towers and the line from the eastern tower to the poacher makes an angle of 75 degrees with the line between the two towers. How far from the western tower is the poacher?



25. The angle of elevation from the edge of a moat around a castle to the top of the castle is $\frac{5\pi}{12}$. When you move 400 yards back from the moat, the angle of elevation of the top of the castle is $\frac{\pi}{3}$.

a. How far is it from the edge of the moat to the top of the castle?

b. How wide is the moat?

c. How high is the castle?

26. If f has an inverse function and $f(2) = 5$, what is $f^{-1}(5)$?

27. In each of the following, find f^{-1}, including its domain.

a. $f(x) = \frac{x+1}{x-5}$.

b. $f(x) = 2x^2 - 8x + 1$, domain of f equal to $[5, \infty)$.

28. If $\tan\theta = -5$, find $\tan(2\theta)$. If, in addition, $\frac{\pi}{2} < \theta < \pi$, find $\sin\theta$ and $\cos\theta$.

29. Write $(f \circ g)$ and $(g \circ f)$ as tables.

x	-1	0	$\frac{1}{2}$	1
$f(x)$	$\frac{1}{2}$	0	1	1

x	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$g(x)$	0	1	2	3

30. If you are on the edge of a merry-go-round of radius 18 inches and your (linear) speed is 10 feet per second, at what speed is the merry-go-round revolving?

31 . Find all solutions of

$$4 \sin x \cos x = 1.$$

32 . Find all solutions of

$$2 \cos^2 \theta - \cos \theta = 1.$$

33. Graph the function for one period and find the amplitude, period, frequency, and phase shift of the graph. Find where the graph is increasing, where it is decreasing, and find the maximum and minimum values.

a.

$$f(x) = 3 \cos(2x + \frac{\pi}{4}).$$

b.

$$f(x) = 1 - \sin(\frac{x}{2} - \frac{\pi}{4}).$$

34. Graph each of the following for one period.

a. $f(x) = \tan(x - \frac{\pi}{4})$.

b. $f(x) = 1 + \csc x$.

c. $f(x) = 2 - \sec x$.

35. Suppose hysteria is a linear function of sleep. If 6 hours of sleep produces 2 hysterias and 4.5 hours of sleep produces 3.75 hysterias,

a. how many hysterias will be produced by 1 hour of sleep, and

b. how much should you sleep to get 5 hysterias?

36. Write each of the following as a sum, then simplify.

a. $\cos(\frac{13\pi}{12}) \cos(\frac{5\pi}{12})$.

b. $\cos(\theta - \frac{\pi}{6}) \sin(\frac{\pi}{3} - \theta)$.

c. $\sin(2\theta + \frac{\pi}{4}) \sin(\theta - \frac{\pi}{4})$.

37. Write each of the following as a product, and simplify if possible.

a. $\sin(\frac{5\pi}{9}) - \sin(\frac{4\pi}{9})$.

b. $\cos(2\theta) + \cos(\theta)$.

38. Evaluate each of the following.

a. $\arctan(-1)$.

b. $\arcsin(-\frac{1}{2})$.

c. $\arccos(-1)$.

d. $\tan(\arctan(19))$.

e. $\arcsin(\sin(\frac{17\pi}{18}))$.

f. $\sin(\arctan(-2))$.

g. $\tan(\arccos(x))$.

39. Write $k(x) \equiv \sin(\log_2(1+x^2))$ as a composition of three simpler functions.

40. If $f(x) = \sin x$ and $g(x) = 2^x$, find, and simplify if possible:

a. $(g \circ f)(\frac{\pi}{2})$ b. $f(\pi g(-2))$ c. $f(2z + \frac{\pi}{6})$ d. $\log_2(f(\frac{\pi}{4}))$ e. $g(\log_4(\theta))$.

HOMEWORK VI

1. Solve $7 - 2|x+4| \geq 1.$

2. Solve $x^2 - 5x < 24.$

3. Solve and graph $\frac{(x^2+9)(x+1)^2 x}{x^2-x-6} \geq 0.$

4. Complete the square: $y = -\frac{1}{2}x^2 + x - 9.$

5. Suppose f is given by

x	-2	0	1	3	4
$f(x)$	1	2	0	2	0

Represent f as a blob diagram, write down (as a set of ordered pairs) the graph of f, and draw the graph of f. Find the domain and range of f, and all intercepts of the graph of f.

6. Find the domain of

(a) $f(x) = \sqrt{x^2 + x - 6}$;

(b) $f(x) = \frac{x}{x^2-2x-8}$.

7. Find the range of $f(x) = x^2 - 3x + 1$.

8. Suppose sweat is a linear function of temperature. If 70 degrees produces one ounce of sweat and 90 degrees produces four ounces of sweat,

a. how much sweat is produced by 150 degrees; and

b. what temperature will produce ten ounces of sweat?

9. Find the linear function f such that $f(-1) = 4$ and $f(3) = 1$.

10. Find the linear function g such that $g^{-1}(-1) = 2$ and $g^{-1}(2) = -2$.

11. Find the linear function whose graph goes through $(1,5)$ and is perpendicular to $3x - 2y = 19$.

12. If $f(x) = -2x^2 + x - 1$, find the maximum or minimum value of f, where the maximum or minimum occurs, where the graph of f is increasing and where it is decreasing.

13. Find the average rate of change, and simplify if possible, of

a. $f(x) = \frac{1}{\sqrt{x}}$ on $[a, a+h]$;

b. $g(x) = 3x^2 - 2x + 9$ on $[a, b]$.

14. Write $f(x) = \log_2(\sin x)$ as the composition of two simpler functions; that is, find h and g so that $f = h \circ g$.

15. Find $(f \circ g)$ and $(g \circ f)$, if the graph of f is $\{(-1,2),(1,0),(2,0)\}$ and the graph of g is $\{(-2,2),(-1,0),(0,-1),(1,-2)\}$. Find f^{-1} or assert that an inverse function for f does not exist; find g^{-1} or assert that an inverse function for g does not exist.

16. If $f(x) = x^2 - 1$ and $g(z) = \sin z$, find a. $f(-2)$. b. $g(\frac{\pi}{4})$ c. $(f \circ g)(\frac{\pi}{6})$. d. $f(\theta)$. e. $f(3\theta)$.

17. In each of the following, find f^{-1} or assert that an inverse function does not exist.

(a) $f(x) = 3x + 5$.

(b) $f(x) = \frac{x+2}{3x-4}$.

(c) $f(x) = x^2 - 4x + 9$; domain of f equal to $(0, \infty)$.

(d) $f(x) = x^2 - 4x + 9$; domain of f equal to $(3, \infty)$.

18. Graph each of the following. For hyperbolas, find asymptotes. For parabolas, find vertices and intercepts. For circles, find the center and radius.

(a) $x^2 + 4x + y^2 - 2y = 1$.

(b) $4x^2 + 8x + y^2 - 6y = 3$.

(c) $y^2 - 6y - 4x^2 - 8x = 11$.

(d) $y = 2x^2 - 12x + 4$.

(e) $y = 5 - |x + 1|$.

(f) $y = \sqrt{x - 2}$.

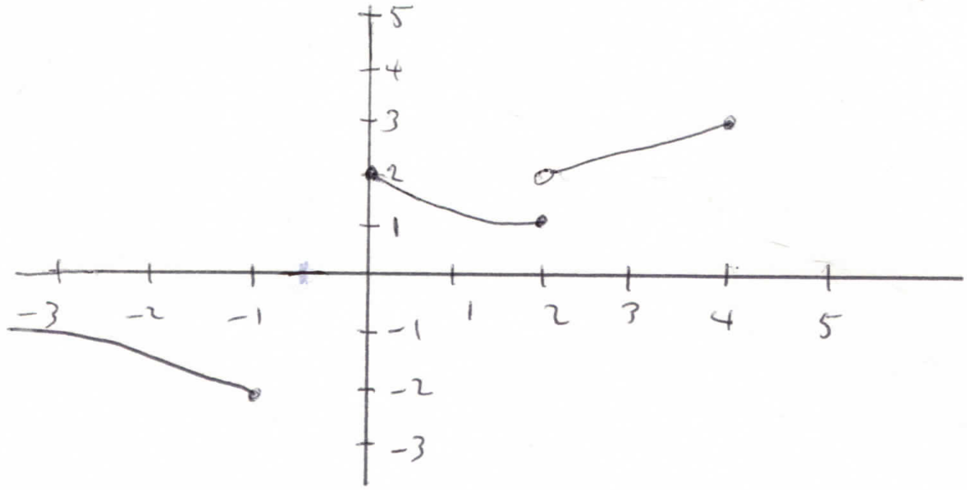
(g) $y = x^3 - 1$.

19. Find a linear function f such that $f(-2) = 3$ and the graph of f has a slope of 4.

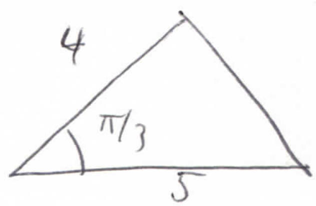
20. If $f(x) = x^2$ and $g(x) = \cos x$, find, and simplify if possible:
a. $f(\theta)$ b. $(f \circ g)(\frac{\pi}{4})$ c. $(g \circ f)(\frac{\pi}{4})$ d. $\frac{1}{g(\frac{\pi}{4})}$ e. $f(z^3\sqrt{z})$ f. $g(\arctan(f(x)))$.

21. Find the area of a sector of radius five feet and angle $\frac{3\pi}{7}$ radians.

22. Use the graph of f below to estimate the domain and range of f, any intercepts of f, where f is increasing or decreasing, any maximum or minimum of f and where it occurs, $f(2)$, and the average rate of change of f over $[0, 2]$.



23. Find the area of the following triangle, where the angle is measured in radians and the sides are measured in meters.



24. Evaluate each of the following.

(a) $\sin(\frac{21\pi}{2})$.

(b) $\tan(-\frac{29\pi}{6})$.

(c) $\cos(\frac{3\pi}{8})$.

(d) $\tan(\frac{7\pi}{12})$.

(e) $\cos(\alpha + \beta)$, if $\sin(\alpha) = \frac{2}{3}, \sin(\beta) = \frac{3}{7}$, $\frac{\pi}{2} < \alpha < \pi$ and $0 < \beta < \frac{\pi}{2}$.

(f) $\sin(2\alpha)$, if $\sin(\alpha) = -\frac{1}{5}$ and $\pi < \alpha < \frac{3\pi}{2}$.

(g) tan, sin and cos of $\frac{\alpha}{2}, \alpha$, and 2α, if $\tan(\alpha) = -7$ and $\pi < \alpha < 2\pi$.

25. The angle of elevation of the top of the spire of a church from the ground ten feet away from the base of the spire is $\frac{5\pi}{12}$. How tall is the spire?

26. Suppose two boats leave the same point at the same time. The first boat travels sixty degrees west of north at ten miles per hour and the second boat travels thirty degrees north of east at twelve miles per hour. How far apart are the boats after two hours?

27. Suppose the angle of elevation of the top of a castle from the ground at the edge of a moat is $\frac{\pi}{3}$. 1,000 yards further away from the castle, the angle of elevation of the top of the castle from the ground is $\frac{\pi}{4}$. Find

a. the distance from the edge of the moat to the top of the castle;

b. the width of the moat; and

c. the height of the castle.

28. Write $\cos(\theta + \frac{\pi}{6})\cos(\frac{\pi}{3} + \theta)$ as a sum, then simplify if possible.

29. Write $\sin(3\theta) - \sin(\theta)$ as a product, then simplify if possible.

30. Evaluate each of the following.

a. $\sin^{-1}(-\frac{1}{2})$.

b. $\arctan(1)$.

c. $\sin(\arccos(\frac{2}{5}))$.

31. Suppose a population triples every 2 hours and there are five of them now.

a. How many will there be after 15 hours?

b. How long will it take for the population to reach 1000? Simplify your answer if possible.

32. Suppose a substance has a half-life of 3 seconds and there is 10 grams of it now.

a. How much will there be 10 seconds from now?

b. How long will it take for the substance to shrink to .1 gram? Simplify your answer if possible.

33. Suppose a fossil contains .01 percent of the C-14 that it contained when it died. How old is the fossil? Simplify your answer if possible. (C-14 has a half-life of 5570 years.)

34. Solve each of the following. Simplify your answer if possible.

a. $\log_3(4x) = 2$.

b. $15 - 2(3^{-2t}) = 5$.

c. $2\log_4(x) - \log_4(x-1) = 1$.

d. $\log_3((x+1)) + \log_3((x+2)) = 2$.

e. $4\ln(\frac{z}{3}) = 8$.

35. Graph $f(x) = -2\sin(\frac{x}{2} + \frac{\pi}{6})$ for one period. Find the amplitude, period, frequency and phase shift.

36. Graph each of the following.

a. $f(x) = \sec(x - \frac{\pi}{4})$ (one period).

b. $f(x) = -\tan x$ (one period).

c. $f(x) = 1 - \csc x$ (one period).

d. $f(x) = \arctan x - \frac{\pi}{2}$.

e. $f(x) = 1 - 2^{-x}$.

f. $f(x) = 1 + \log_3(x)$.

ANSWERS TO CHAPTER I HOMEWORK

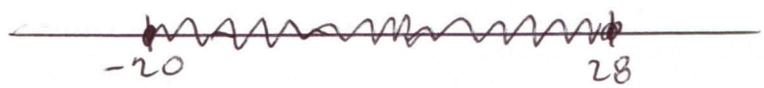
1. 9. **2.** $\frac{7}{2}$. **3.** $\frac{2}{5}$. **4.** $\frac{14}{5}$. **5.** 7.

6. a. $[\frac{16}{5}, \infty)$; also known as $x \geq \frac{16}{5}$.

b. $x > \frac{1}{2}$ or $x < -\frac{9}{2}$.

c. no solutions.

d. $[-20, 28]$; also known as $-20 \leq x \leq 28$.



7. $(-8, -2)$; also known as $-8 < x < -2$.

8. a. $(3z - 2x^2)(9z^2 + 6zx^2 + 4x^4)$ (this is a difference of two cubes $(a^3 - b^3)$, with $a = 3z, b = 2x^2$).

b. $(2x - 3)^2$.

c. $(y - 4)(y + 1)$.

d. $(2x - 3)(x + 1)$.

e. Irreducible over the integers.

9. a. 8 b. $3^{6-4\sqrt{2}}$ c. 3^6 d. 2 e. $(2 + \sqrt{2})$
f. $\sqrt{a}$ g. $a^3\sqrt{a}$ or $a^{\frac{7}{2}}$ h. $(a\sqrt{a} - 3)$.

10. a. $(x + \frac{9}{2})^2 - \frac{85}{4}$.

b. $2(y - 5)^2 - 41$.

11. a. $(x - 4)^2 = 26 \rightarrow x = 4 \pm \sqrt{26}$.

b. $(y - 3)^2 = \frac{26}{3} \rightarrow y = 3 \pm \sqrt{\frac{26}{3}}$.

12. $(x - 9)(x + 2) = 0 \rightarrow x = 9$ or -2.

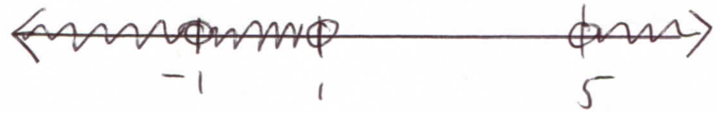
13. $y = \frac{-7 \pm \sqrt{57}}{4}$.

14. a. one real solution b. no real solutions.

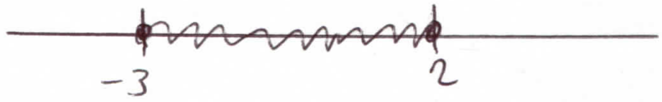
15. $(x-1)(x+3)(x-2)$.

16. $x^2 + \frac{1}{x+1}$.

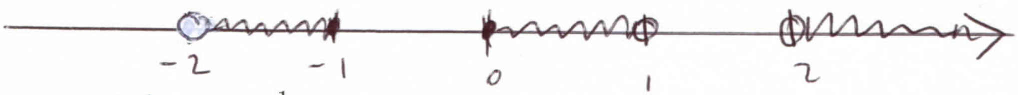
17. a. $\{z \,|\, z < -1, -1 < z < 1 \text{ or } z > 5\}$.



b. $[-3, 2]$.



c. $\{y \,|\, -2 < y \leq -1 \text{ or } 0 \leq y < 1 \text{ or } y > 2\}$.



18. a. $x = 2$ b. $x = -\frac{1}{2}$.

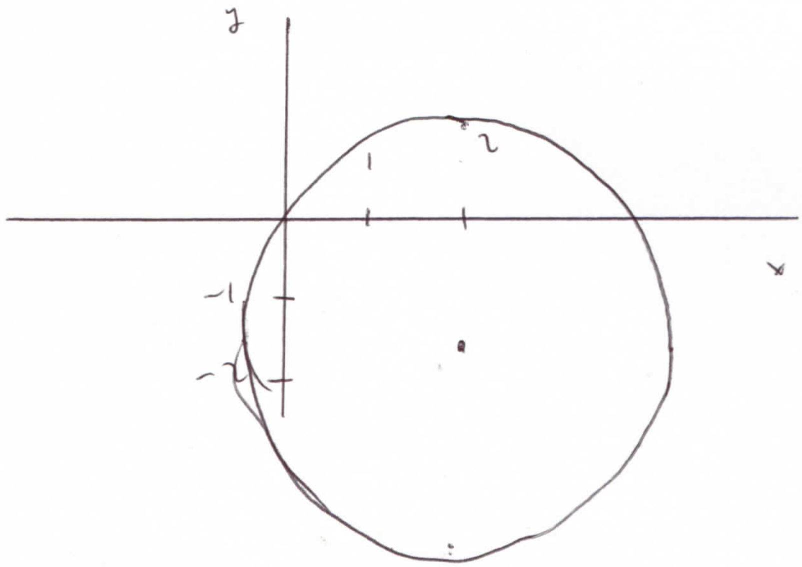
19. distance is $5\sqrt{2}$; midpoint is $(-\frac{3}{2}, \frac{1}{2})$.

20. $a = 6, b = 5$.

21. My speed is $\frac{220}{3}$ miles per hour; I'll catch the bus at **3 : 45** p.m.

22. $(x-1)^2 + (y-4)^2 = 4$.

23. center is $(2, -\frac{3}{2})$, radius is $\frac{3\sqrt{3}}{2}$.



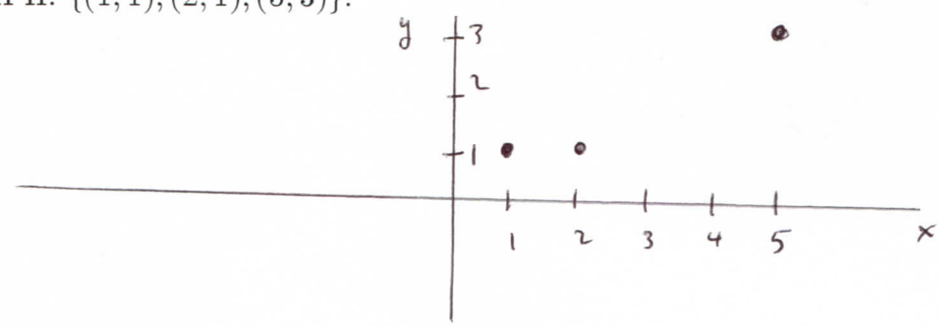
ANSWERS TO CHAPTER II HOMEWORK

1. a. 9 b. 8 c. 64 d. $g(4) = 18$ e. undefined. f. z^2
g. y^6 h. $h(1) = -1$ i. $h(-2)$ is undefined j. $g(-1) = -7$ k. $(5x^2 - 2)$ ℓ. $(25y^2 - 20y + 4)$.

2. TABLE:

x	1	2	5
$f(x)$	1	1	3

GRAPH: $\{(1,1), (2,1), (5,3)\}$.

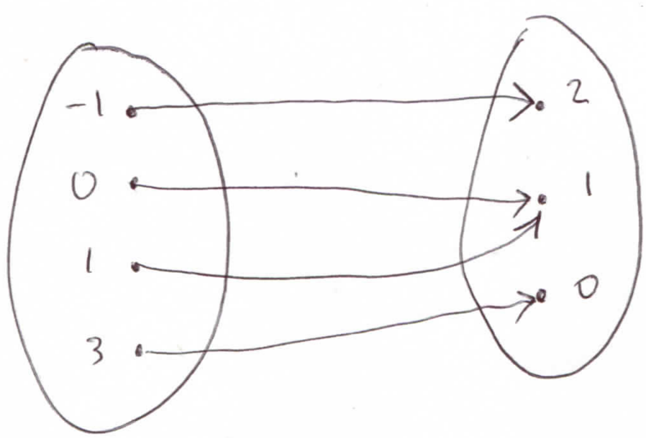


The domain is $\{1, 2, 5\}$; the range is $\{1, 3\}$.

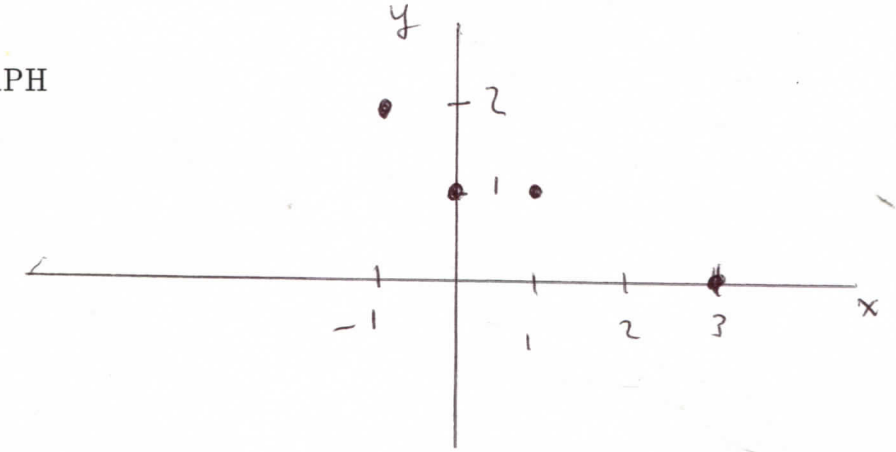
3. TABLE:

x	-1	0	1	3
$f(x)$	2	1	1	0

BLOB DIAGRAM



GRAPH



The domain is $\{-1, 0, 1, 3\}$. The range is $\{0, 1, 2\}$. $f(0) = 1$.

4. a. is a function, b. isn't. (Both by the vertical line test.)

5. a. y intercept is 5; no x intercept.
b. y intercept is -2; x intercepts are $\frac{1}{4}(1 \pm \sqrt{17})$.
c. y intercept is -1; x intercept is $-\frac{1}{2}$.

6. a. $\{x \,|\, x < -1 \text{ or } 1 < x < 3 \text{ or } x > 3\}$.
b. $\{x \,|\, -3 \leq x < 2 \text{ or } 2 < x \leq 5 \text{ or } x > 6\}$.

7. a. $[-\frac{1}{8}, \infty)$.
b. $\{y \,|\, y \neq -\frac{1}{3}\}$.

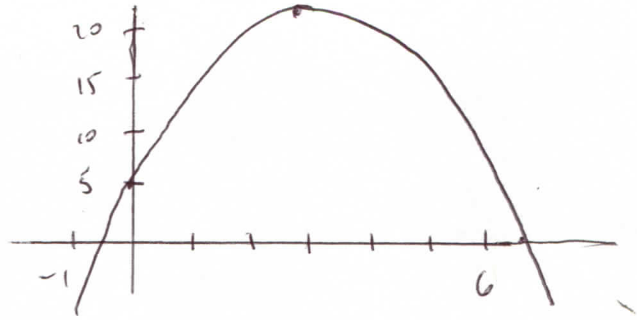
8. domain looks like $[-2, 6]$; range looks like $[-1, 4]$. The x intercepts appear to be 2 and 4; the y intercept appears to be 2.

9. a. $(-\frac{5}{2}, \frac{3}{2})$.
b. $z = \frac{15}{2}$.

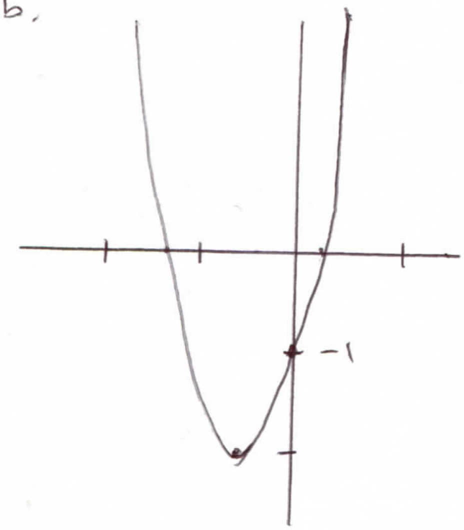
10. a. $f(x) = 19x + 16$.
b. $f(x) = 1 - x$.
c. $f(x) = \frac{1}{2}x + \frac{5}{2}$.
d. $f(x) = -2x$.
e. $f(x) = 2$.

11. a. vertex is $(3, 23)$; the x intercepts are $3 \pm \sqrt{\frac{23}{2}}$; the y intercept is 5.
b. vertex is $(-\frac{1}{2}, -2)$; the x intercepts are $\frac{1}{2}(-1 \pm \sqrt{2})$; the y intercept is -1.
c. vertex is $(\frac{1}{4}, -\frac{1}{16})$; the x intercepts are 0 and $\frac{1}{2}$; the y intercept is 0.

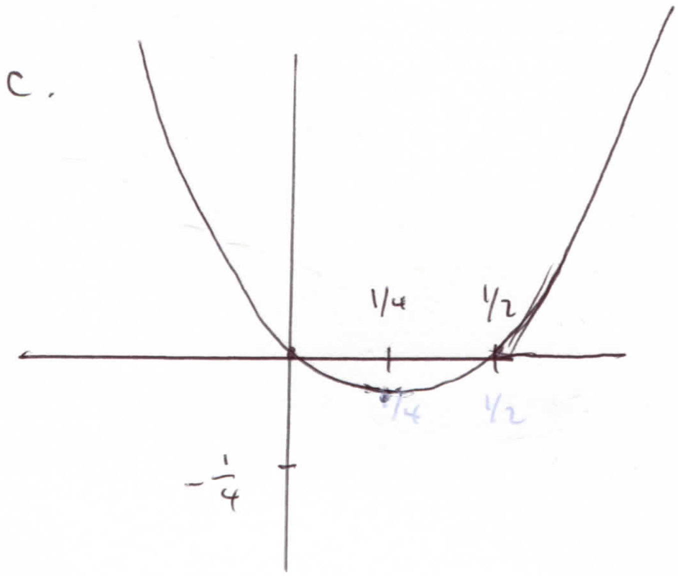
GRAPHS: a.



b.



c.



12. $f(x) = -\frac{1}{2}x + \frac{5}{2}$.

13. $g(x) = 3x + 11$.

14. If $C(F) \equiv$ degrees Celsius equal to F degrees Fahrenheit, then $C(F) = \frac{5}{9}(F - 32)$. (use given facts: $C(32) = 0, C(212) = 100$, so that we are looking for a function whose graph is a line through $(32, 0)$ and $(212, 100)$.)

15. 115 pounds (if $f(x) \equiv$ pounds of body fat from x grams of candy, then f is linear, $f(1) = 10$ and $f(3) = 40$, thus $f(x) = 40 + 15(x - 3)$.)

16. a. $\frac{50}{3}$ centimeters cubed b. $\frac{13}{5}$ liters (if $g(w) \equiv$ centimeters cubed of plant produced by w liters of water, then $g(w) = 20 + \frac{10}{3}(w - 5)$.)

17. a. maximum of 23, occurring at $x = 3$; no minimum; decreasing on $(3, \infty)$.

b. minimum of -2, occurring at $x = -\frac{1}{2}$; no maximum; decreasing on $(-\infty, -\frac{1}{2})$.

c. minimum of $-\frac{1}{16}$ occurring at $x = \frac{1}{4}$; no maximum; decreasing on $(-\infty, \frac{1}{4})$.

18. The desired numbers are 5 and 10. (Let $f(x) \equiv xy$, where $2x + y = 20$, so that $y = 20 - 2x$ and $f(x) = x(20 - 2x)$; find x that maximizes $f(x)$, as in number 1.)

19. a. $(5 + \frac{5}{2}\sqrt{5})$ seconds.

b. 5 seconds

c. 500 feet.

20. 50 feet by 100 feet (if $A(x) \equiv$ area of pasture when the side perpendicular to the barn is x feet, then $A(x) = x(200 - 2x)$; maximize this the way we always maximize quadratic functions.)

21. a. $3(b+a)-2$ b. -1 c. $-\frac{1}{ab}$ d. 1; 4 e. $-\frac{1}{\sqrt{ab}(\sqrt{a}+\sqrt{b})}$ f. $2a+h$.

22. 6 yards per second.

23. $\frac{5}{3}$.

24. The domain is $\{x \mid -4 \le x < -2, -1 < x < 2, \text{ or } 2 < x \le 4\}$. The range is $(-\infty, 2]$. The x intercept is 1 and the y intercept is 1. The function has a maximum of 2, occurring at $x = -3$. It has no minimum. It is increasing on $(-4,-3), (-1,0)$ and $(2,4)$. It is decreasing on $(-3,-2)$ and $(0,2)$.

25. $(f \circ g)(x) = \sqrt{1-x^2}$, with domain $[-1,1]$. $(g \circ f)(x) = 1 - x$, with domain $[0,\infty)$.

26. a. 2 b. 4 c. 4 d. $2x^2+2$ e. $2x^2+8x+8$.

27. $h = f \circ g$, where $f(x) = \frac{1}{\sqrt{x}}, g(x) = 37x^3 + 19x$.

28. The graph of $(f \circ g)$ is $\{(0,0),(1,2)\}$. The graph of $(g \circ f)$ is $\{(-1,1),(1,1)\}$.

29.

x	0
$(f \circ g)(x)$	3

x	-1	1	2
$(g \circ f)(x)$	0	0	3

30. $(f \circ g)(-1) \sim 2; (f^{-1})(2) \sim 1; (g \circ f)(0) \sim 0$.

31.

x	0	2	-1	5	4
$f^{-1}(x)$	-1	0	1	3	5

32. $\{(3,-2),(1,-1),(2,0),(-1,2)\}$.

33. 1.

34. $(2-\sqrt{14})$.

35. $\frac{11}{3}$.

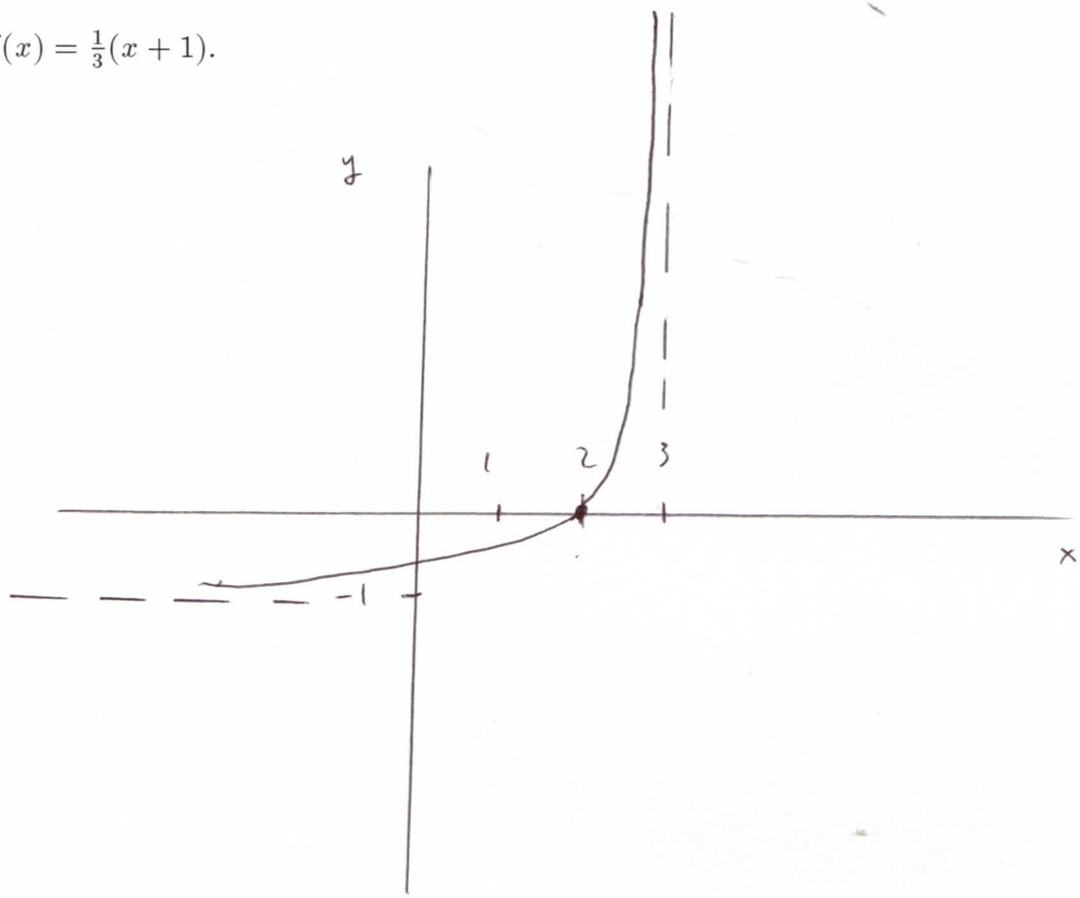
36. a. $f^{-1}(y) = \frac{5-y}{2}$.
b. $f^{-1}(y) = -1 + \sqrt{y-8}$. (domain is $[9,\infty)$)

c. $f^{-1}(y) = \frac{4-y}{2y+3}$.

37. Only c. and d. have inverse functions.

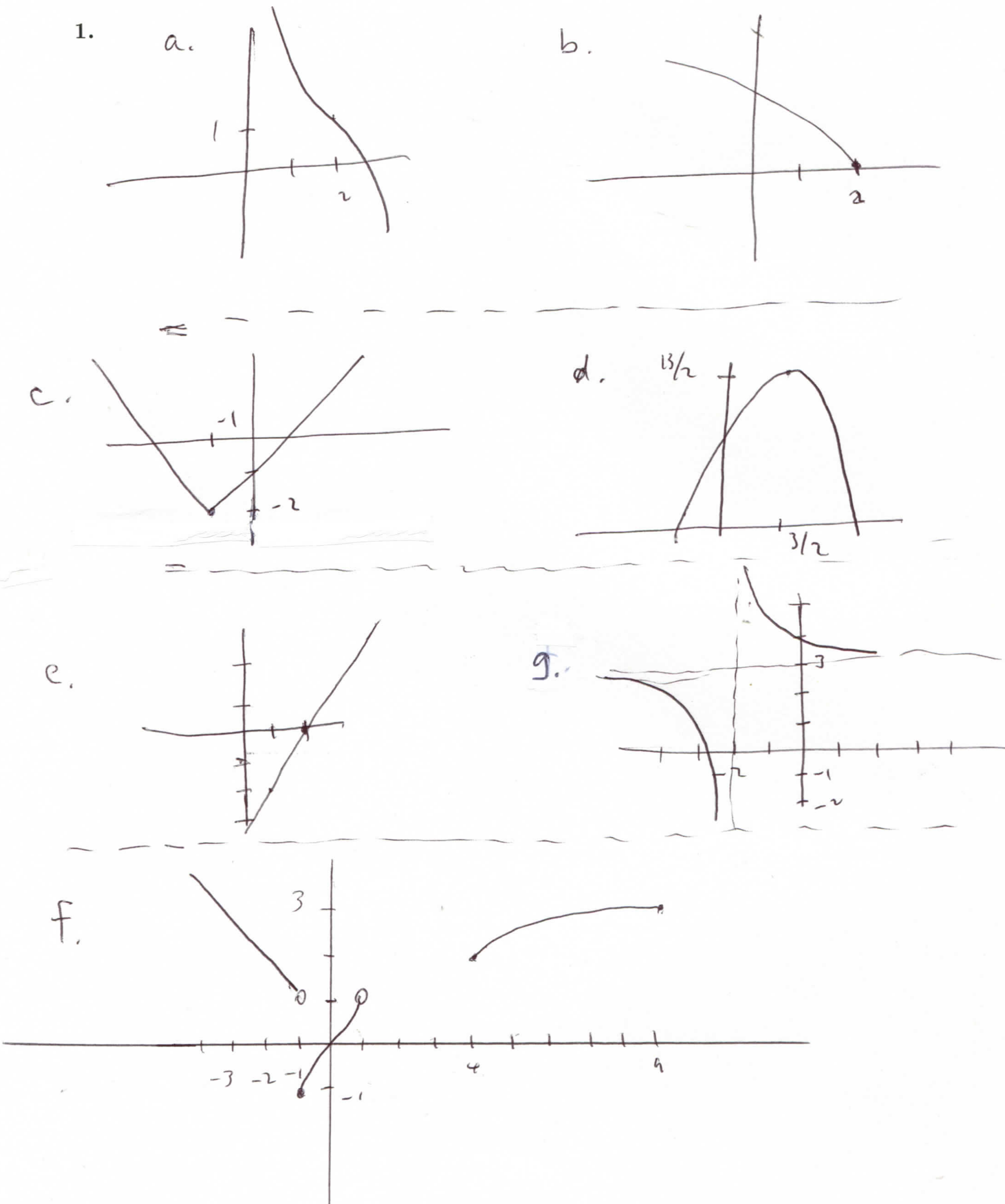
38. $f(x) = \frac{1}{3}(x+1)$.

39.



ANSWERS TO CHAPTER III HOMEWORK

1.



2. a. y intercept is 3, x intercept is $1 + 2^{\frac{1}{3}}$.

b. y intercept is $\sqrt{2}$, x intercept is 2.

c. y intercept is -1; x intercepts are 1 and -3.

d. y intercept is 2; x intercepts are $\frac{1}{2}(3 \pm \sqrt{13})$.

e. y intercept is -4; x intercept is 2.

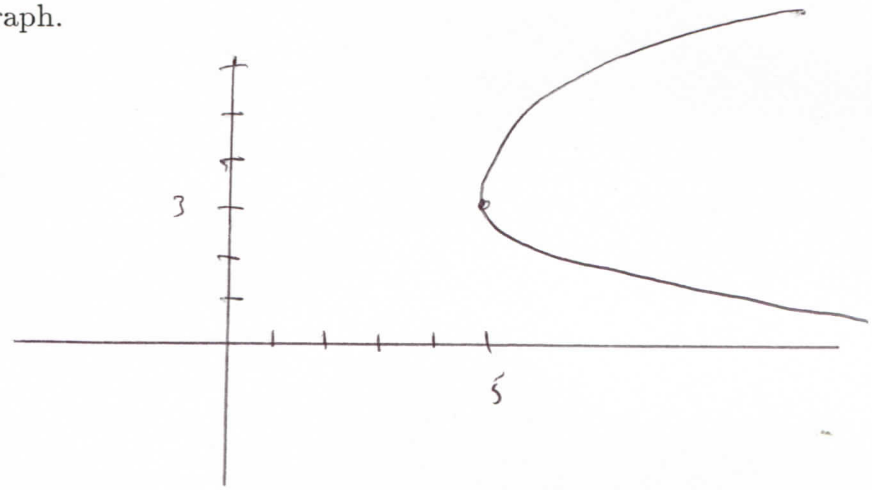
f. y intercept is 0; x intercept is 0.

g. y intercept is $\frac{7}{2}$; x intercept is $-\frac{7}{3}$.

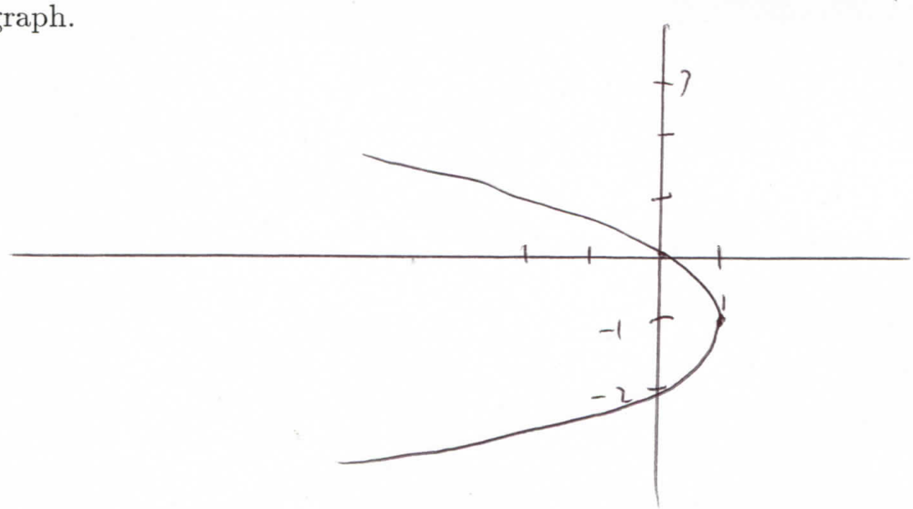
3. a. no symmetry.

b. symmetric about the origin; not symmetric about the x axis or y axis.

4. a. This equation is $(x-5) = 2(y-3)^2$, a parabola with a vertex of $(5,3)$; here is the graph.



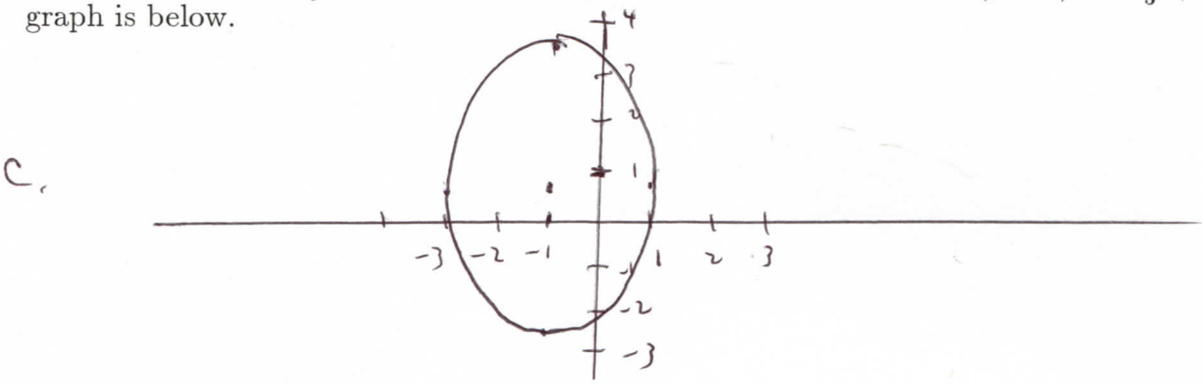
b. This equation is $(x-1) = -(y+1)^2$, a parabola with a vertex of $(1,-1)$; here is the graph.

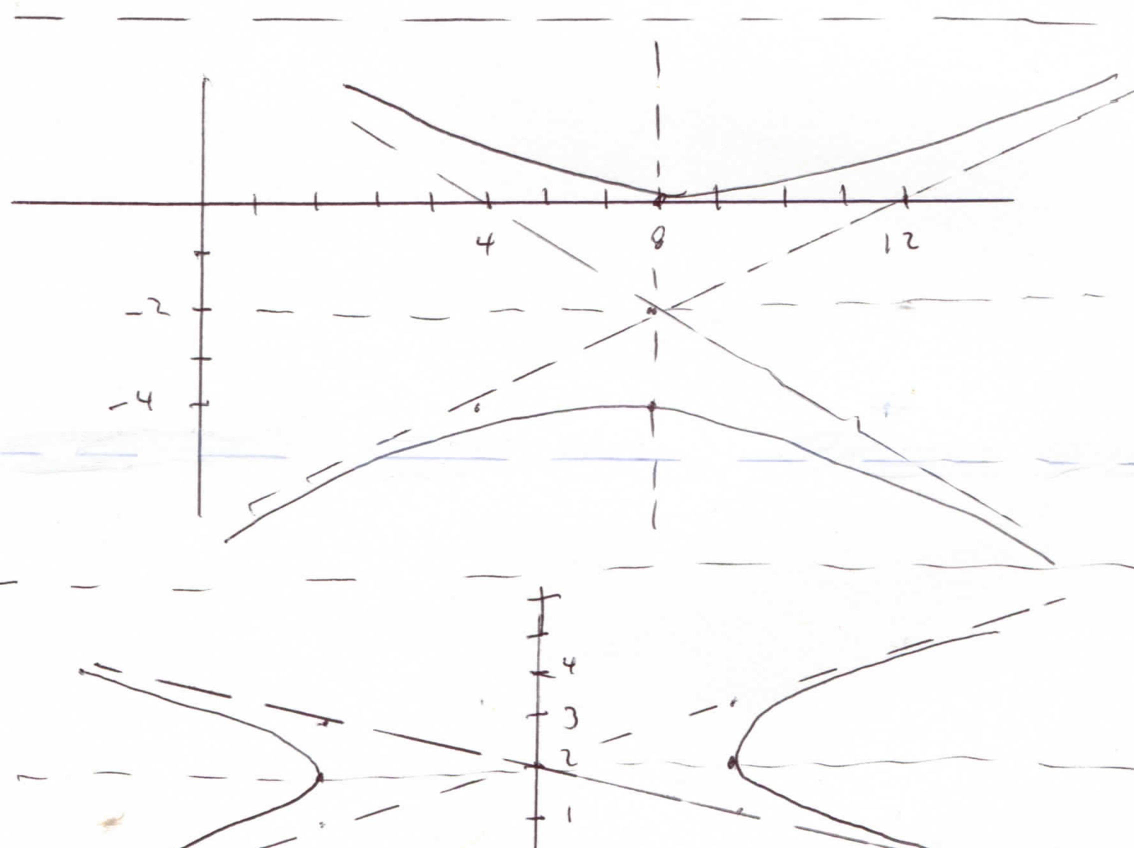


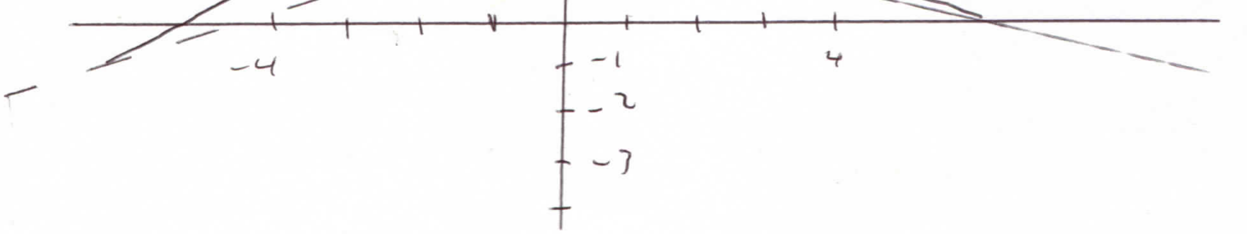
c. This equation is $\frac{(x+1)^2}{4} + \frac{(y-\frac{1}{2})^2}{9} = 1$, graph is below.

d. This equation is $\frac{(y+2)^2}{4} - \frac{(x-8)^2}{16} = 1$, a hyperbola with asymptotes $(y+2) = \pm\frac{1}{2}(x-8)$; graph is below.

e. This equation is $\frac{x^2}{9} - (y-2)^2 = 1$, a hyperbola with asymptotes $(y-2) = \pm\frac{1}{3}x$; graph is below.





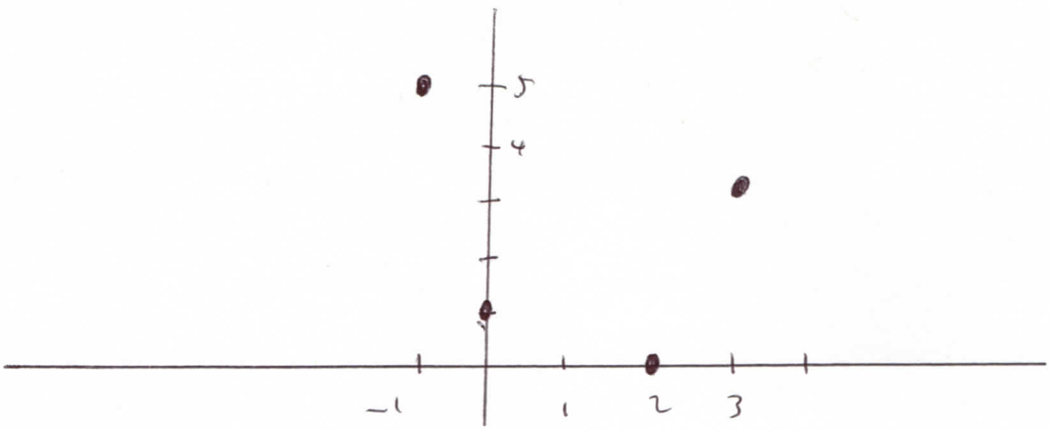


5. a. $\left(30,000 + \frac{625,000}{13}\right)$ dollars.

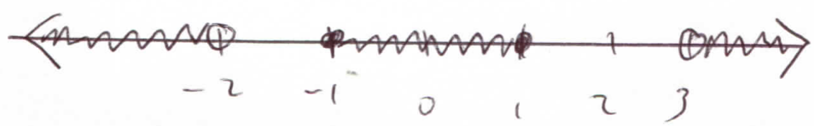
b. 10.5 hours.

(For both a. and b., let $f(x)$ be your annual salary twenty years from now when you spend x hours per week on homework this quarter. Then f is linear, $f(5) = 30,000$ and $f(18) = 55,000$; the graph of f is a straight line through $(5, 30,000)$ and $(18, 55,000)$. Part a. wants $f(30)$; part b. wants x such that $f(x) = 40,000$.)

6. Domain is $\{-1, 0, 2, 3\}$. Range is $\{5, 1, 0, 3\}$. The x intercept is 2 and the y intercept is 1. Graph is $\{(-1,5), (0,1), (2,0), (3,3)\}$.



7. $\{x \,|\, x < -2, -1 \leq x \leq 1 \text{ or } x > 3\}$.



8. 7.

9. $\{(3,-1), (1,0), (2,1)\}$.

10. a. z^2 b. $\frac{1}{\theta+1}$ c. $\frac{1}{(y^2+1)^2}$ d. $\frac{1}{2}$ e. $\frac{x}{1+x}$ f. $\frac{r+1}{r+2}$.

11. $f(2) = -1, f^{-1}(2) = -3$, the x-intercept of the graph of f equals the y-intercept of the graph of f^{-1} equals -1, and the y-intercept of the graph of f equals the x-intercept of the graph of f^{-1} equals $-\frac{1}{2}$.

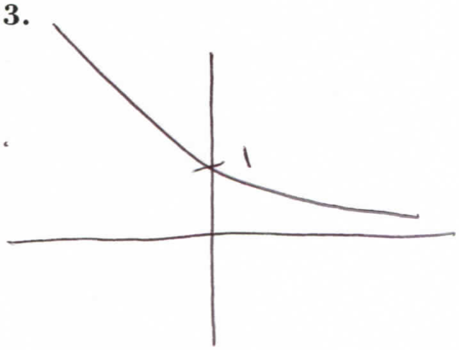
ANSWERS TO CHAPTER IV HOMEWORK

1. $50(2)^{24}$.

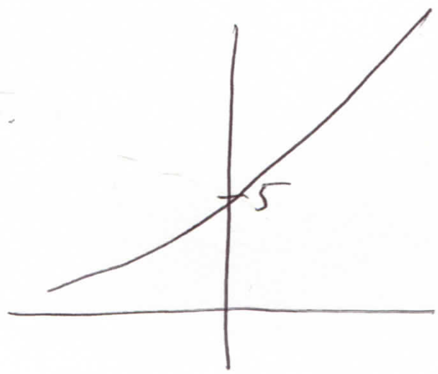
2. $100(2)^{-1800}$.

3.

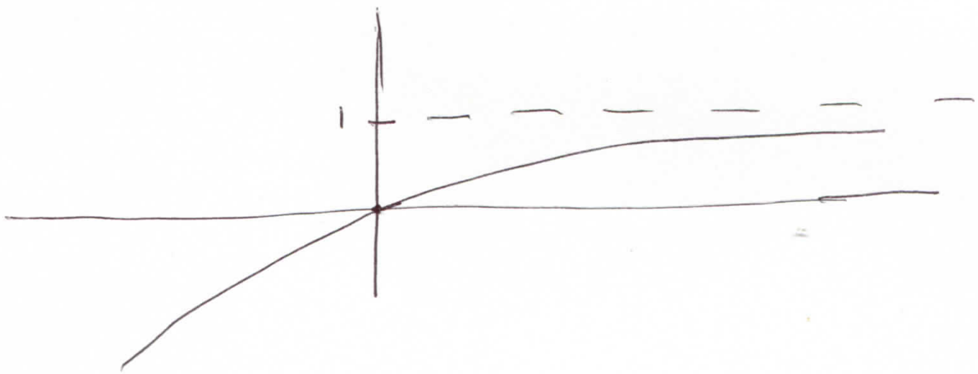
a.



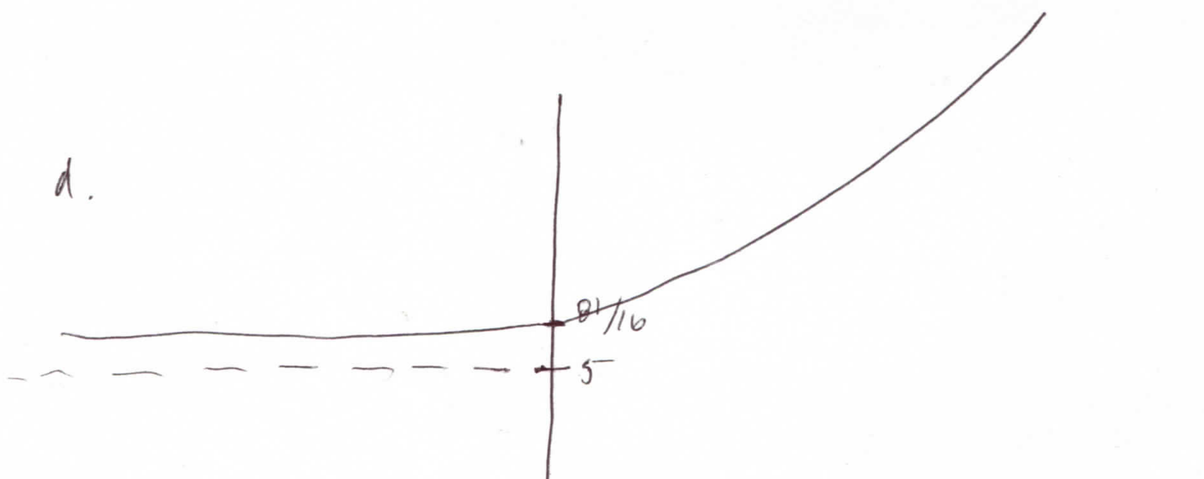
b.



c.



d.



4. $200(3)^{\frac{73}{10}}$.

5. $5(2)^{\frac{45}{4}}$ pounds.

6. a. $x = -\frac{3}{2}$.
b. $t = \frac{3}{2}$.

7. $h = f \circ g$, where $f(x) = 2^x, g(x) = x^2$.

8. $(f \circ g)(x) = \frac{1}{2}\left(2^{x^2}\right)$; $(g \circ f)(x) = \left(1 - \left(\frac{1}{x}\right)^2\right)$.

9.

x	5	0	1	2
$f^{-1}(x)$	-1	0	2	3

10. a. 3 b. Not defined. c. $-\frac{1}{2}$.

11. -3.

12. a. $x = \frac{5}{2}$. b. $z = -\frac{5}{4}$. c. $y = \log_2 6$. d. $x = \pm\sqrt{31}$. e. $t = 0$. f. $w = 2$. g. $x = \pm 1$. h. $x = 1 \pm \sqrt{2}$. i. $t = \frac{1}{2}(1 + \ln 2)$. j. $x = 2(1 + \sqrt{2})$. k. $x = \frac{1}{2}\ln 6$.

13. a. $\frac{9}{2}$. b. -2. c. $\frac{1}{2}$. d. $\frac{1}{2}$.

14. $50\log_2(5,000)$ days.

15. $8\log_2 5$ years.

16. $16\log_2 5$ years.

17. $11,140\log_2(10)$ years olds.

18. a. $\frac{2^z}{z}$ b. $(2^t - t)$ c. x d. 8 e. 1.

19. $[-7, -3]$.

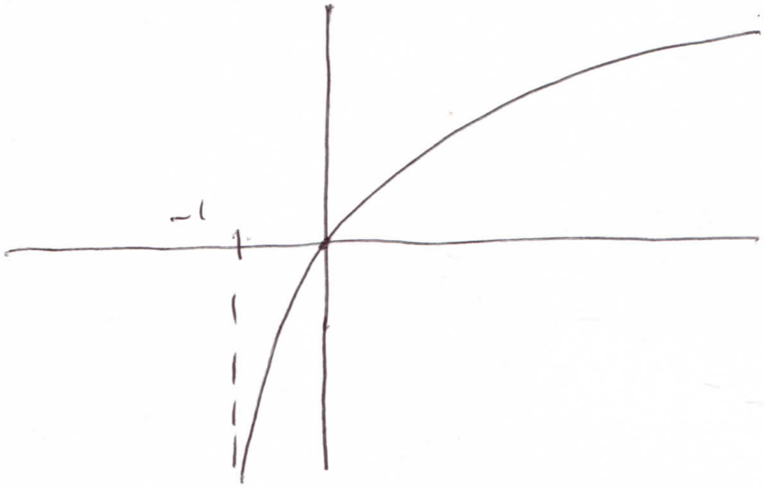
20. $\{x \mid x \leq 2 \text{ or } x > 5\}$.

21. $y = \frac{1}{4}(x + 6)^2 - 11$.

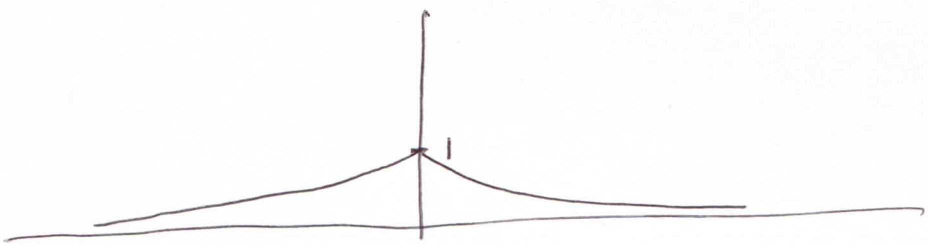
22. a. $\frac{5(2^h - 1)}{h}$.
b. $2^a\left[\frac{5(2^h-1)}{h}\right]$.
c. $5(2a + h)$.
d. $\frac{3}{7}$.

23.

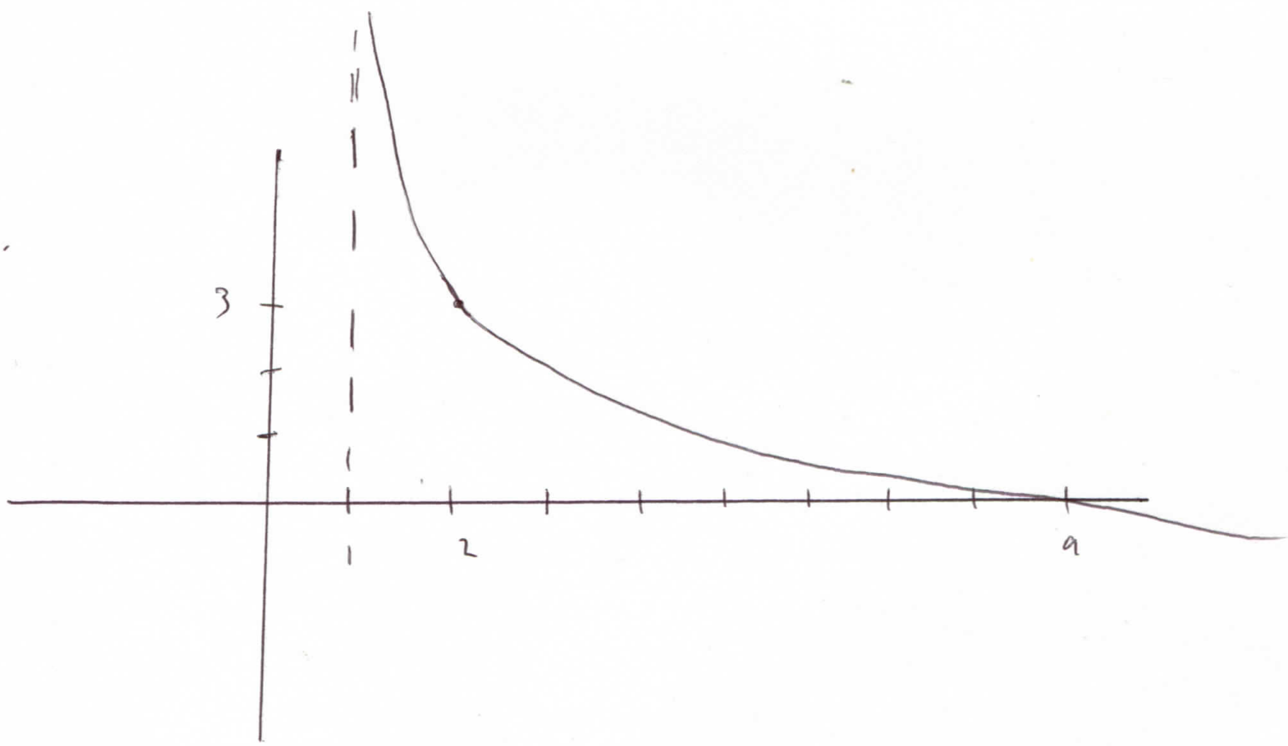
a.



b.



c.



ANSWERS TO CHAPTER V HOMEWORK

1. $\frac{29\pi}{18}$ radians. **2.** 315 degrees. **3.** $\frac{5}{6}$ radians. **4.** $\frac{40}{3\pi}$ feet.

5. 132,000 radians per hour; or 2,200 radians per minute.

6. 400π feet per minute. **7.** 150 revolutions per minute.

8. $\frac{3\pi}{4}$ feet squared. **9.** $-\frac{\sqrt{45}}{7}$.

10.

$$\sin(-\frac{\pi}{2}) = -1, \quad \cos(-\frac{\pi}{2}) = 0, \quad \sin(\frac{77\pi}{2}) = 1, \quad \cos(\frac{72\pi}{2}) = 0,$$

$$\sin(91\pi) = 0, \quad \cos(91\pi) = -1, \quad \sin(-\frac{77\pi}{2}) = -1, \quad \cos(-\frac{77\pi}{2}) = 0,$$

$$\sin(-4\pi) = 0, \quad \cos(-4\pi) = 1, \quad \sin(\frac{19\pi}{2}) = -1, \quad \cos(\frac{19\pi}{2}) = 0.$$

11. f only. **12.** $\{x \mid x < -2 \text{ or } x > 4\}$.

13. a. $-\frac{1}{\sqrt{2}}$ b. $\frac{1}{2}$ c. -1 d. $-\frac{1}{\sqrt{5}}$ e. $\frac{1}{26}$ f. -2

g. $\frac{\sqrt{3}-1}{\sqrt{3}+1}$ (using sum-of-angles); *or*, equivalently, $\frac{1}{2+\sqrt{3}}$ (using half-angle)

h. -1 i. $\frac{5\sqrt{3}+2}{12}$

j. $\frac{-(\sqrt{3}+1)}{2\sqrt{2}}$ (using sum-of-angles); *or*, equivalently, $-\frac{1}{2}\sqrt{2+\sqrt{3}}$ (using half-angle)

k. $\frac{1}{5}$ l. $\frac{2\sqrt{6}-\sqrt{3}}{10}$ m. $\frac{1}{2+\sqrt{3}}$ (using half-angle)

n. $-\sqrt{\frac{1}{2}(1-\frac{1}{\sqrt{2}})}$ (using half-angle).

14. $\sin(\alpha) = -\frac{3}{\sqrt{10}}$; $\cos(\alpha) = \frac{1}{\sqrt{10}}$; $\tan(\frac{\alpha}{2}) = \frac{-3}{1+\sqrt{10}}$;

$$\sin(\frac{\alpha}{2}) = -\sqrt{\frac{1}{2}(1-\frac{1}{\sqrt{10}})}; \cos(\frac{\alpha}{2}) = \sqrt{\frac{1}{2}(1+\frac{1}{\sqrt{10}})};$$

$\tan(2\alpha) = \frac{3}{4}$; $\sin(2\alpha) = -\frac{3}{5}$; $\cos(2\alpha) = -\frac{4}{5}$.

15. $y = \sin x$ is symmetric about the origin only; $y = \cos x$ is symmetric about the y-axis only; $y = \tan x$ is symmetric about the origin only

16. vertex: $(\frac{3}{2}, -\frac{1}{4})$; x-intercepts: none; y-intercept: -1; maximum: $-\frac{1}{4}$ at $x = \frac{3}{2}$;

increasing on $(-\infty, \frac{3}{2})$; decreasing on $(\frac{3}{2}, \infty)$; range: $(-\infty, -\frac{1}{4}]$. SEE p. 133 for graph

17. a. hypotenuse is $\frac{6\sqrt{2}}{(\sqrt{3}+1)}$; remaining side is $\frac{3(\sqrt{3}-1)}{(\sqrt{3}+1)}$.

b. side opposite $\frac{\pi}{4}$ is $\frac{8}{\sqrt{6}}$; remaining side is $\frac{4(\sqrt{3}+1)}{\sqrt{6}}$.

18. $3\sqrt{3}$. **19.** $\frac{40\sqrt{2}}{(\sqrt{3}+1)}$ feet.

20. The height of the pigeon above the ground is $2\sqrt{3}$ feet.

21. a. $\frac{1}{2\sqrt{(1-\frac{1}{\sqrt{2}})}} = \frac{1}{2}\sqrt{2+\sqrt{2}}$ miles. (This is $\frac{1}{2\sqrt{2}\sin(\frac{\pi}{8})}$.)

b. $\frac{\sqrt{\frac{1}{2}(1+\frac{1}{\sqrt{2}})}}{2\sqrt{(1-\frac{1}{\sqrt{2}})}} = \frac{1}{2}(1+\frac{1}{\sqrt{2}})$ miles. (This is $\sin(\frac{3\pi}{8})$ times the answer in a.)

22. a. $100\sqrt{19}$ feet. b. $250\sqrt{3}$ feet.

23. a.

$$40\sqrt{61 - 60\cos(\frac{\pi}{8})} = 40\sqrt{61 - 60\sqrt{\frac{1}{2}(1+\frac{1}{\sqrt{2}})}}\ \text{feet}\ .$$

b. $240\sin(\frac{3\pi}{8}) = 240\sqrt{\frac{1}{2}(1+\frac{1}{\sqrt{2}})}$ feet.

24. $3\sqrt{\frac{1}{3}(2+\sqrt{3})}$ miles.

25. a. $\frac{400\sqrt{3}}{\sqrt{(2-\sqrt{3})}} = 400\sqrt{3}\sqrt{2+\sqrt{3}}$ yards.

b. $200\sqrt{3}$ yards.

c. $\frac{200(3+\sqrt{3})}{(\sqrt{3}-1)}$ (using sum-of-angles) or equivalently (using half-angles)

$$200\sqrt{3}\sqrt{\frac{\sqrt{3}}{(2-\sqrt{3})}} = 200\sqrt{3}\sqrt{\sqrt{3}(2+\sqrt{3})}\ \text{yards}\ .$$

26. 2. **27.** a. $f^{-1}(y) = \frac{1+5y}{y-1}$, domain $\{y \mid y \neq 1\}$. b. $f^{-1}(y) = 2 + \frac{1}{2}\sqrt{2y+14}$, with domain $[11, \infty)$.

28. $\tan(2\theta) = \frac{5}{12}$; $\sin(\theta) = \frac{5}{\sqrt{26}}$; $\cos(\theta) = -\frac{1}{\sqrt{26}}$.

29.

x	-1	0	$\frac{1}{2}$	1
$(g \circ f)(x)$	2	1	3	3

x	$-\frac{1}{2}$	0
$(f \circ g)(x)$	0	1

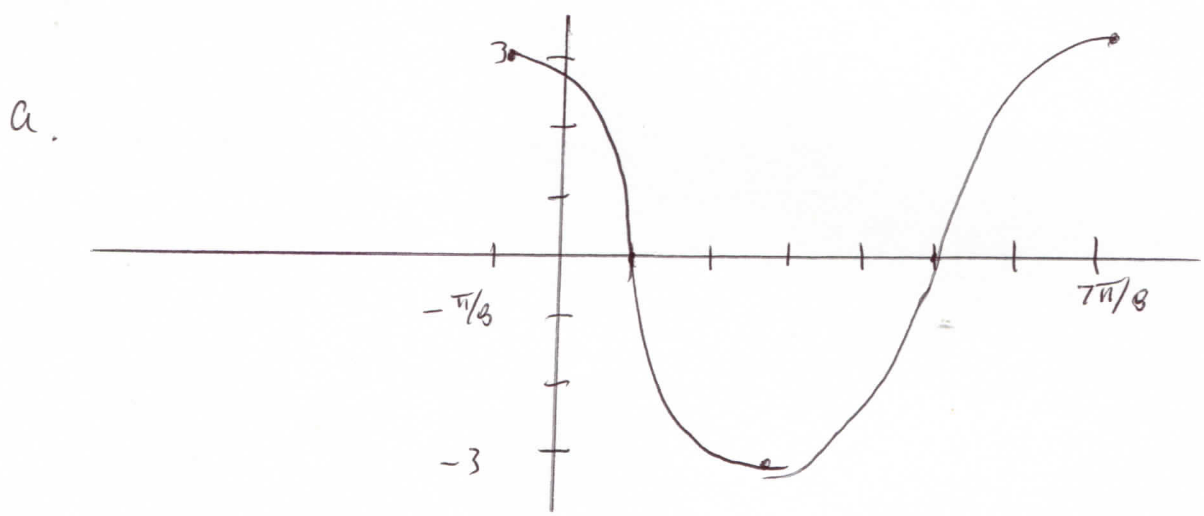
30. $\frac{20}{3}$ radians per second.

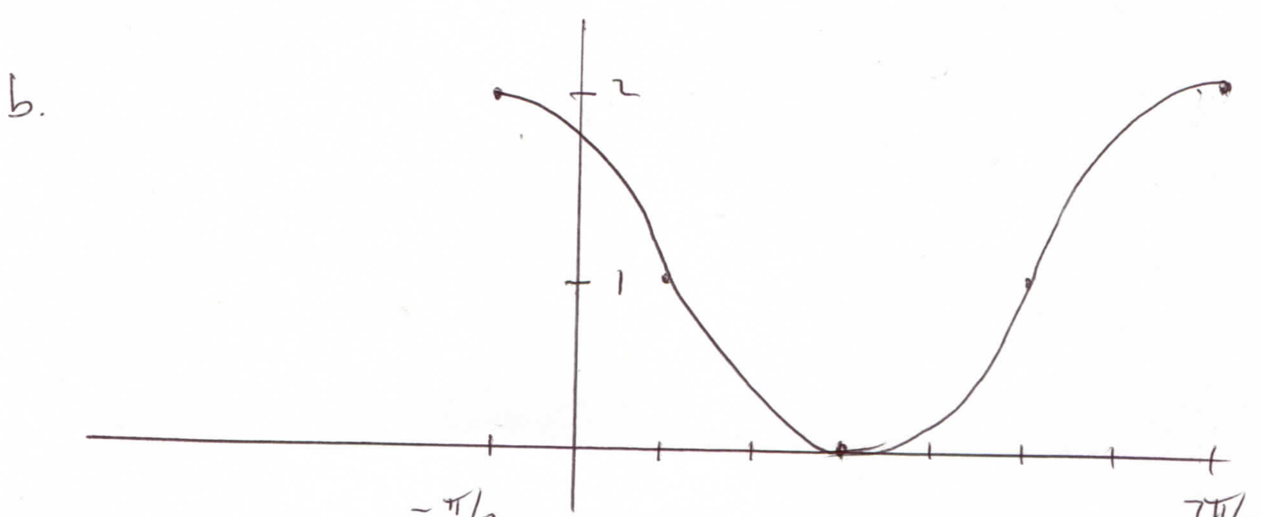
31. $x = \frac{\pi}{12} + k\pi$ or $\frac{5\pi}{12} + k\pi$, for any integer k.

32. $\theta = 2k\pi$ or $2k\pi \pm \frac{2\pi}{3}$, for any integer k.

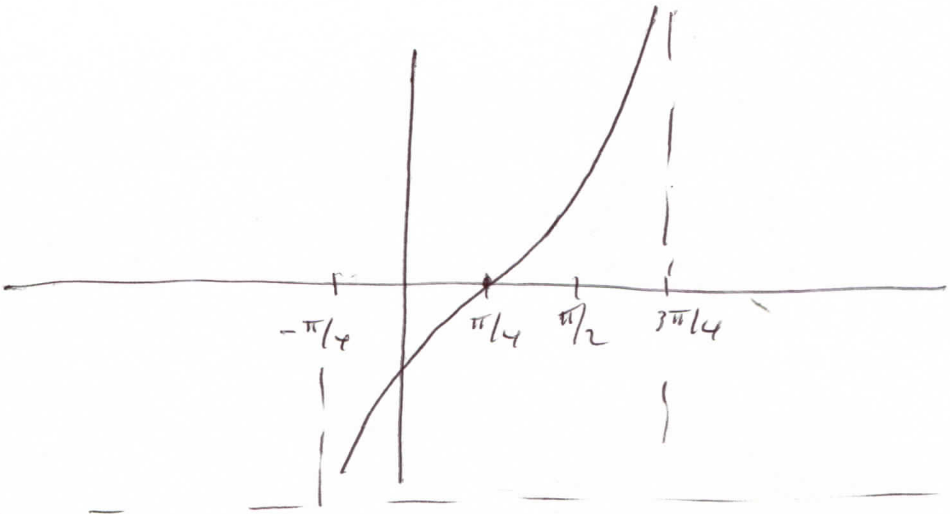
33. a. amplitude is 3; period is π; frequency is $\frac{1}{\pi}$; phase shift is $-\frac{\pi}{8}$. Graph (see below) is increasing on $(\frac{3\pi}{8}, \frac{7\pi}{8})$ and decreasing on $(-\frac{\pi}{8}, \frac{3\pi}{8})$. The function has a maximum of 3 and a minimum of -3.

b. amplitude is 1; period is 4π; frequency is $\frac{1}{4\pi}$; phase shift is $\frac{\pi}{2}$. The graph (see below) is increasing on $(\frac{\pi}{2}, \frac{5\pi}{2})$ and decreasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $(\frac{5\pi}{2}, \frac{7\pi}{2})$. The maximum value of the function is 2 and the minimum value is 0.

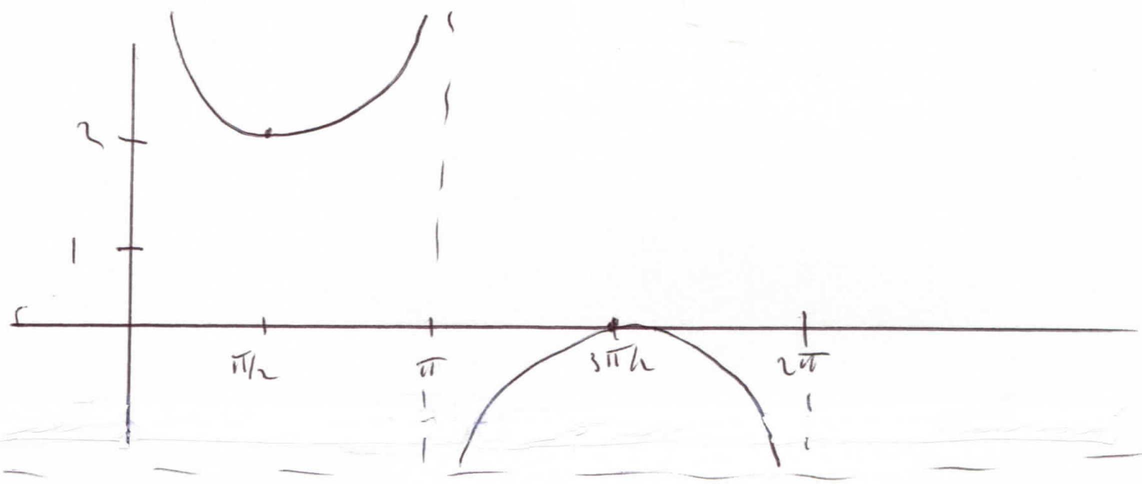




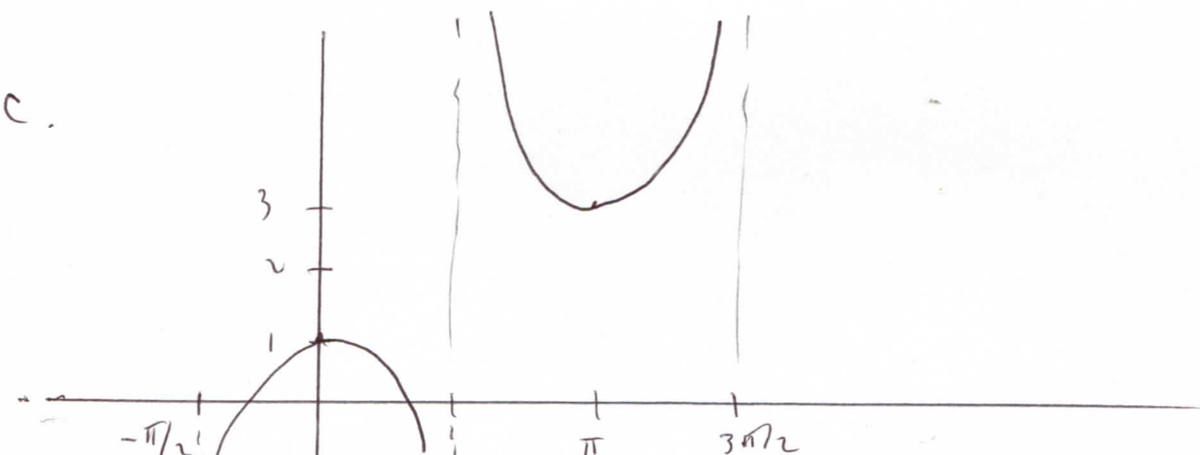
34. a.



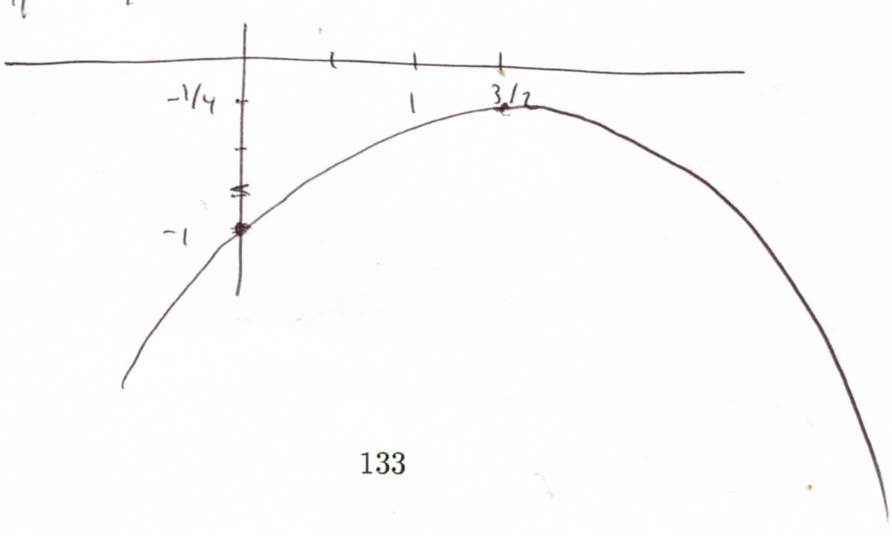
b.



c.



#16.



35. a. $\frac{47}{6}$ hysterias.

b. $\frac{24}{7}$ hours.

36. a. This is

$$\frac{1}{2}\left[\cos(\frac{2\pi}{3})+\cos(\frac{3\pi}{2})\right]=-\frac{1}{4}.$$

b. This is

$$\frac{1}{2}\left[\sin(\frac{\pi}{2}-2\theta)+\sin(\frac{\pi}{6})\right]=\frac{1}{2}\left[\cos(2\theta)+\frac{1}{2}\right].$$

c. This is

$$\frac{1}{2}\left[\cos(\theta+\frac{\pi}{2})-\cos(3\theta)\right]=-\frac{1}{2}\left[\sin(\theta)+\cos(3\theta)\right].$$

37. a. This is

$$2\sin\left(\frac{1}{2}(\frac{\pi}{9})\right)\cos\left(\frac{1}{2}(\pi)\right),$$

which equals zero.

b.

$$2\cos(\frac{3\theta}{2})\cos(\frac{\theta}{2}).$$

38. a. $-\frac{\pi}{4}$.

b. $-\frac{\pi}{6}$.

c. π.

d. 19.

e. This is

$$\arcsin\left(\sin(\pi-\frac{\pi}{18})\right)=\arcsin\left(\sin(\frac{\pi}{18})\right)=\frac{\pi}{18}.$$

f. $-\frac{2}{\sqrt{5}}$. Our right triangle picture tells us that $\sin(\arctan(2))=\frac{2}{\sqrt{5}}$; since $\tan(-\theta)=-\tan(\theta)$, for any θ, $\arctan(-x)=-\arctan(x)$, for any x; thus

$$\sin(\arctan(-2))=\sin(-\arctan(2))=-\sin(\arctan(2))=-\frac{2}{\sqrt{5}}.$$

g.

$$\tan\left(\arccos(x)\right) = \begin{cases} -\frac{\sqrt{1-x^2}}{x} & -1<x<0 \\ \frac{\sqrt{1-x^2}}{x} & 0<x<1. \end{cases}$$

This deserves some explanation. For $0<x<1$, our usual right triangle picture gives us

$$\tan(\arccos(x))=\frac{\sqrt{1-x^2}}{x}.$$

Now suppose $-1 < x < 0$. Since $0 < (-x) < 1$,

$$\tan(\arccos(-x)) = \frac{\sqrt{1-(-x)^2}}{(-x)} = -\frac{\sqrt{1-x^2}}{x}.$$

To show that $\tan(\arccos(-x)) = \tan(\arccos(x))$, let

$$y \equiv \arccos(-x).$$

Then $\cos(y) = -x$ and $0 \leq y \leq \pi$. This implies that $x = \cos(\pi - y)$, so that $\arccos(x) = \pi - y$, and

$$\arccos(-x) \equiv y = \pi - \arccos(x).$$

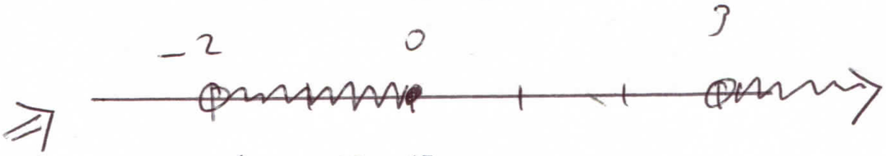
Thus

$$\tan(\arccos(-x)) = \tan(\pi - \arccos(x)) = -\tan(-\arccos(x)) = \tan(\arccos(x)).$$

39. $k = f \circ g \circ h$, where $h(x) = 1 + x^2, g(x) = \log_2 x, f(x) = \sin x$.

40. a. 2 b. $\frac{1}{\sqrt{2}}$ c. $\sqrt{3}(\sin z)(\cos z) + \frac{1}{2}(\cos^2 z - \sin^2 z)$
d. $-\frac{1}{2}$ e. $\sqrt{\theta}$.

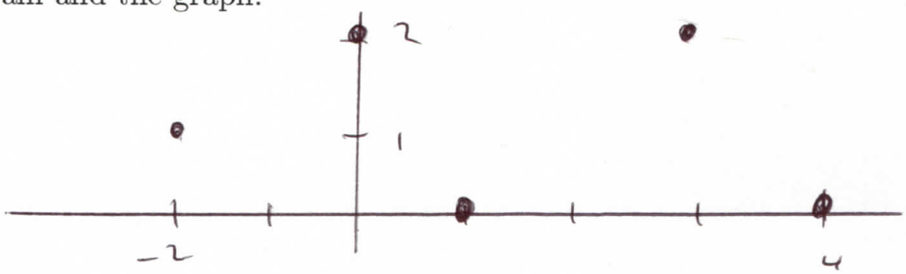
ANSWERS TO HOMEWORK VI



1. $[-7,-1]$. **2.** $(-3,8)$.

3. $\{x \mid -2 < x \leq 0 \text{ or } x > 3\}$. **4.** $y = -\frac{1}{2}(x-1)^2 - \frac{17}{2}$.

5. The graph of f is $\{(-2,1),(0,2),(1,0),(3,2),(4,0)\}$. Domain is $\{-2,0,1,3,4\}$. Range is $\{0,1,2\}$. The x intercepts are 1 and 4; the y intercept is 2. Here is the blob diagram and the graph.



6. a. $\{x \mid x \leq -3 \text{ or } x \geq 2\}$ b. $\{x \mid x \neq 4 \text{ and } x \neq -2\}$.

7. $[-\frac{5}{4},\infty)$. **8.** . 13 ounces. b. 130 degrees.

9. $f(x) = 4 - \frac{3}{4}(x+1)$. **10.** $g(x) = \frac{2}{3}(1-2x)$. **11.** $f(x) = \frac{1}{3}(17-2x)$.

12. The function has a maximum value of $-\frac{7}{8}$, occuring at $x = \frac{1}{4}$; it has no minimum value. The graph is increasing on $(-\infty,\frac{1}{4})$ and decreasing on $(\frac{1}{4},\infty)$.

13. a. $\frac{-1}{\sqrt{a(a+h)}(\sqrt{a}+\sqrt{a+h})}$. b. $3(b+a)-2$.

14. $h(x) = \log_2 x, g(x) = \sin x$.

15. f has no inverse function.

x	-2	0
$(f \circ g)(x)$	0	2

x	1	2
$(g \circ f)(x)$	-1	-1

x	2	0	-1	-2
$g^{-1}(x)$	-2	-1	0	1

16. a. 3. b. $\frac{1}{\sqrt{2}}$. c. $-\frac{3}{4}$. d. (θ^2-1). e. $(9\theta^2-1)$.

17. a. $f^{-1}(y) = \frac{1}{3}(y-5)$ b. $f^{-1}(y) = \frac{2+4y}{3y-1}$

c. No inverse function d. $f^{-1}(y) = 2 + \sqrt{y-5}$.

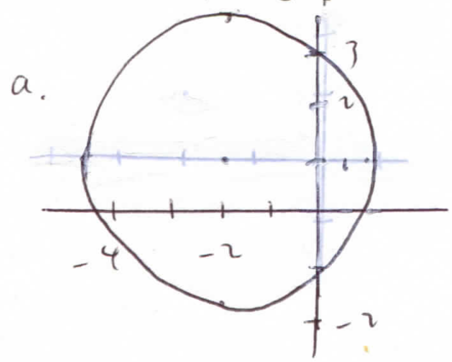
18. a. This is a circle with center $(-2, 1)$, radius $\sqrt{6}$.

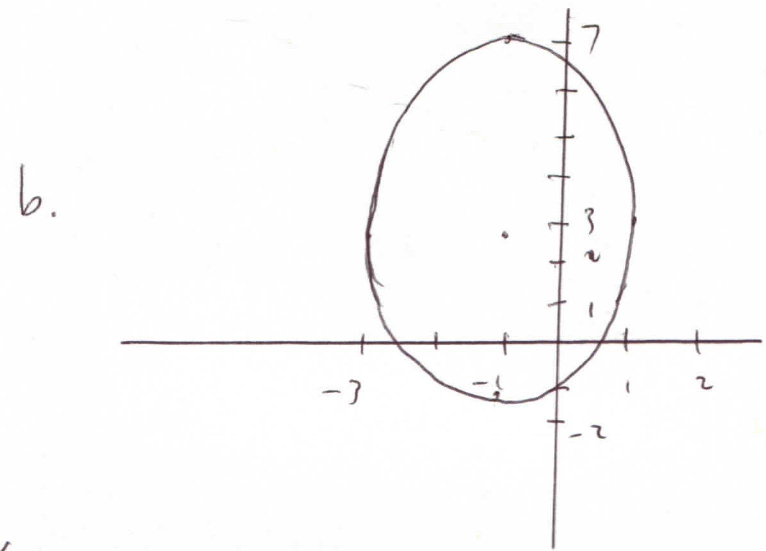
b. This is an ellipse: $\frac{(x+1)^2}{4} + \frac{(y-3)^2}{16} = 1$.

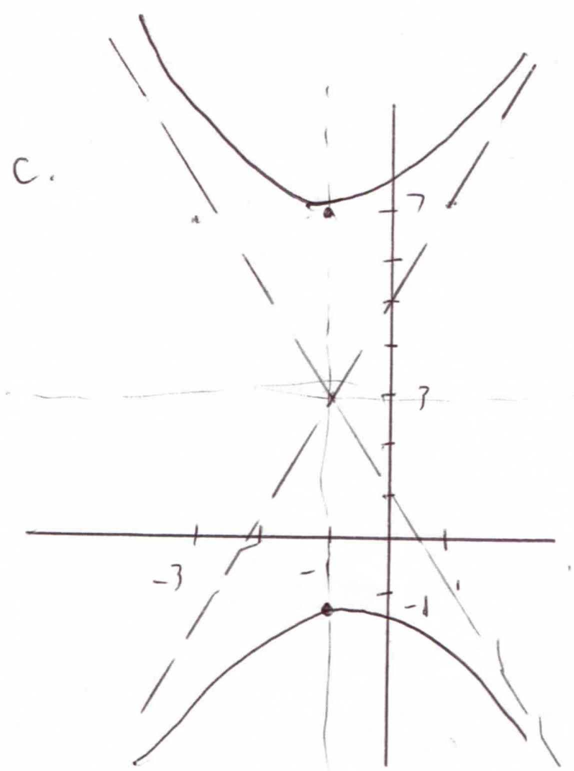
c. This is a hyperbola with asymptotes $(y-3) = \pm 2(x+1)$: $\frac{(y-3)^2}{16} - \frac{(x+1)^2}{4} = 1$.

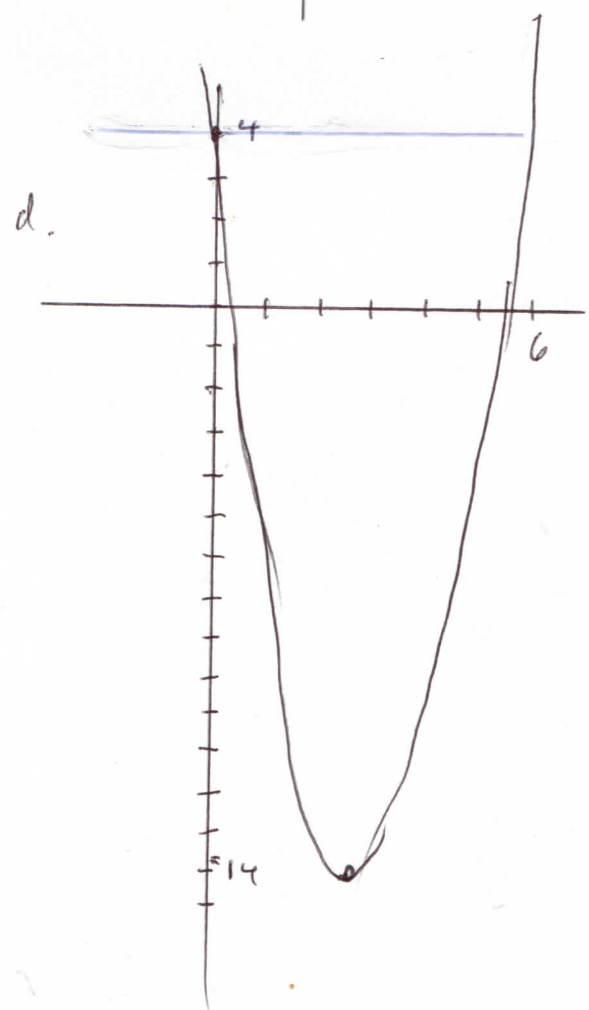
d. This is a parabola with vertex $(3, -14)$, x intercepts $3 \pm \sqrt{7}$, y intercept 4: $(y + 14) = 2(x-3)^2$.

Here are the graphs:

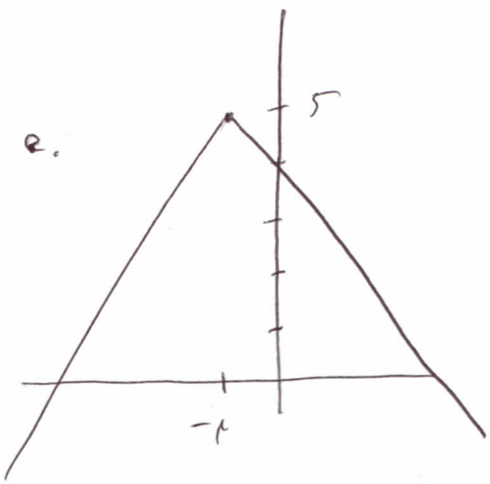


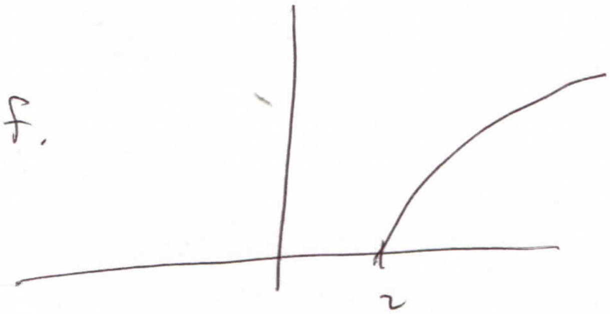


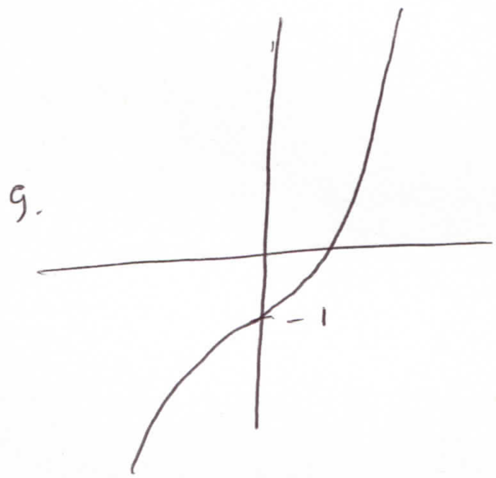




18. continued:







19. $f(x) = 4x + 11$.

20. a. θ^2 b. $\frac{1}{2}$ c. $\cos(\frac{\pi^2}{16})$ d. $\sqrt{2}$ e. z^7 f. $\frac{1}{\sqrt{1+x^4}}$.

21. $\frac{75\pi}{14}$ feet squared.

22. Domain is $\{x \mid x \leq -1 \text{ or } 0 \leq x \leq 4\}$. Range is $\{y \mid -2 \leq y < 0 \text{ or } 1 \leq y \leq 3\}$. The y intercept is 2; there are no x intercepts. The graph is increasing on $(2, 4)$ and decreasing on $(-\infty, -1)$ and $(0, 2)$. The maximum of f is 3, occuring at $x = 4$. The minimum of f is -2, occuring at $x = -1$. $f(2) = 1$. The average rate of change of f over $[0, 2]$ is $-\frac{1}{2}$.

23. $5\sqrt{3}$ meters squared.

24. a. 1 b. $\frac{1}{\sqrt{3}}$ c. $\sqrt{\frac{1}{2}(1 - \frac{1}{\sqrt{2}})}$ d. $\frac{1}{\sqrt{3}-2}$ e. $-\frac{(10\sqrt{2}+6)}{21}$ f. $\frac{4\sqrt{6}}{25}$

g. $\cos(\alpha) = \frac{1}{\sqrt{50}}$; $\sin(\alpha) = -\frac{7}{\sqrt{50}}$; $\tan(\frac{\alpha}{2}) = -\frac{7}{(\sqrt{50}+1)}$; $\sin(\frac{\alpha}{2}) = \sqrt{\frac{1}{2}(1 - \frac{1}{\sqrt{50}})}$; $\cos(\frac{\alpha}{2}) = -\sqrt{\frac{1}{2}(1 + \frac{1}{\sqrt{50}})}$; $\tan(2\alpha) = \frac{7}{24}$; $\sin(2\alpha) = -\frac{7}{25}$; $\cos(2\alpha) = -\frac{24}{25}$.

25. $\frac{10}{2-\sqrt{3}}$ feet. **26.** $4\sqrt{91}$ miles.

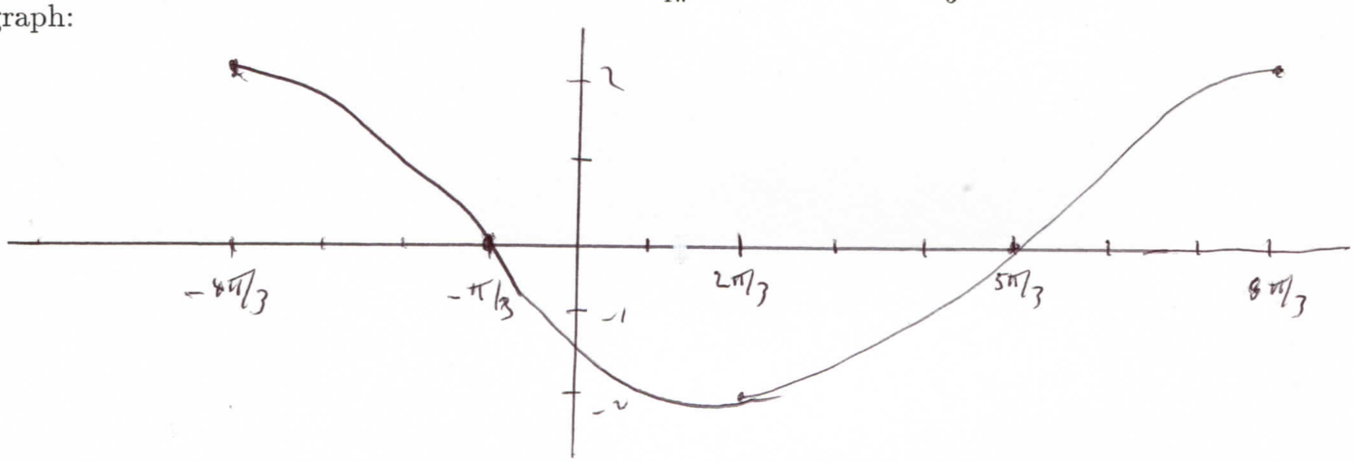
27. a. $\frac{1{,}000}{\sqrt{1-\frac{\sqrt{3}}{2}}}$ yards b. $\frac{500}{\sqrt{1-\frac{\sqrt{3}}{2}}}$ yards c. $\frac{500\sqrt{3}}{\sqrt{1-\frac{\sqrt{3}}{2}}}$ yards.

28. $\frac{1}{2}\left(\frac{\sqrt{3}}{2}-\sin(2\theta)\right)$. **29.** $2(\sin(\theta))(\cos(2\theta))$. **30.** a. $-\frac{\pi}{6}$. b. $\frac{\pi}{4}$. c. $\frac{\sqrt{21}}{5}$.

31. a. $5(3^{\frac{15}{2}})$ b. $2\log_3(200)$ hours. **32.** a. $10(2^{-\frac{10}{3}})$ grams b. $6\log_2(10)$ seconds. **33.** $22{,}280\log_2(10)$ years.

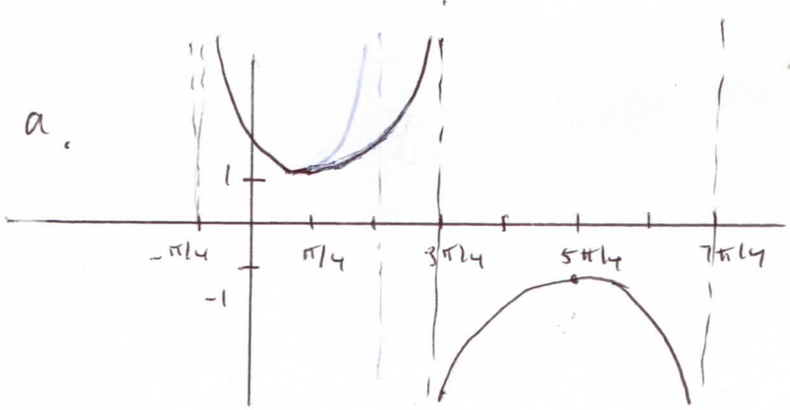
34. a. $x=\frac{9}{4}$. b. $t=-\frac{1}{2}\log_3(5)$. c. $x=2$. d. $x=\frac{1}{2}(\sqrt{37}-3)$. e. $z=3e^2$.

35. amplitude is 2; period is 4π; frequency is $\frac{1}{4\pi}$; phase shift is $-\frac{\pi}{3}$. Here is the graph:

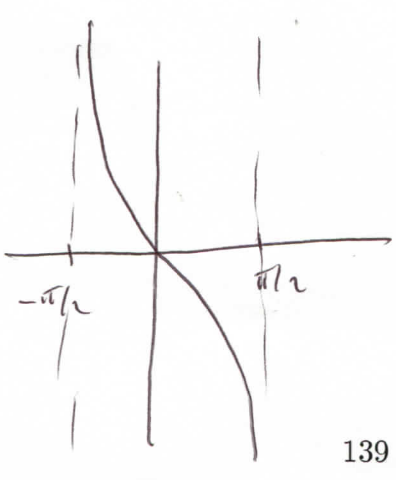


36.

a.

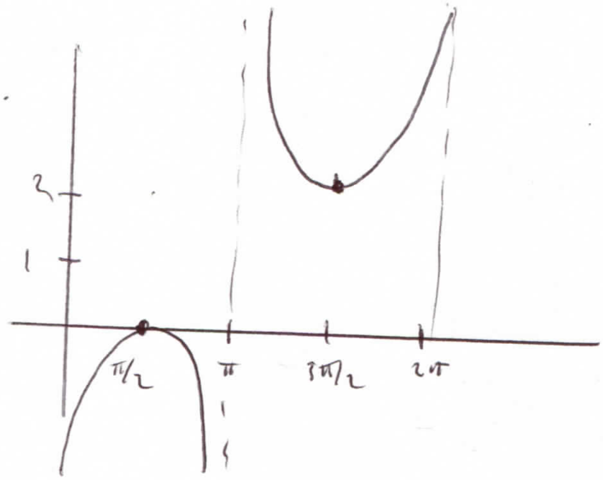


b.

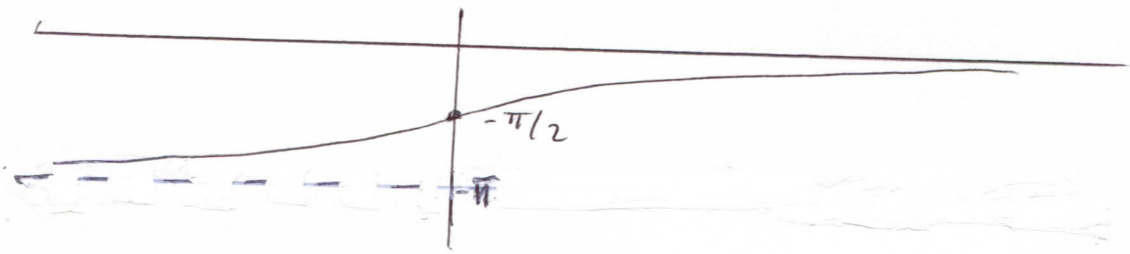


36. continued:

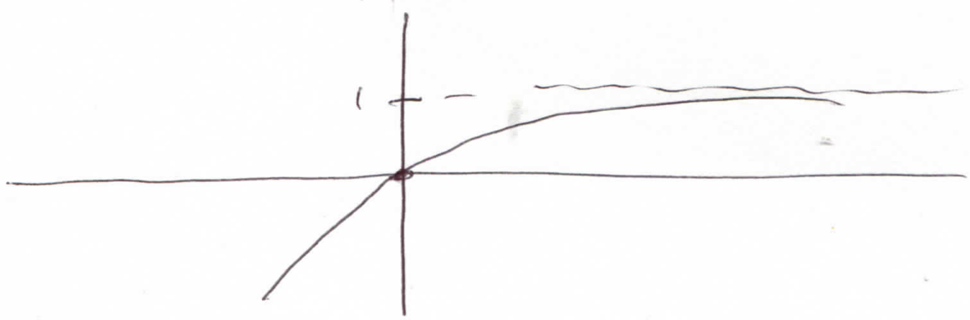
c.



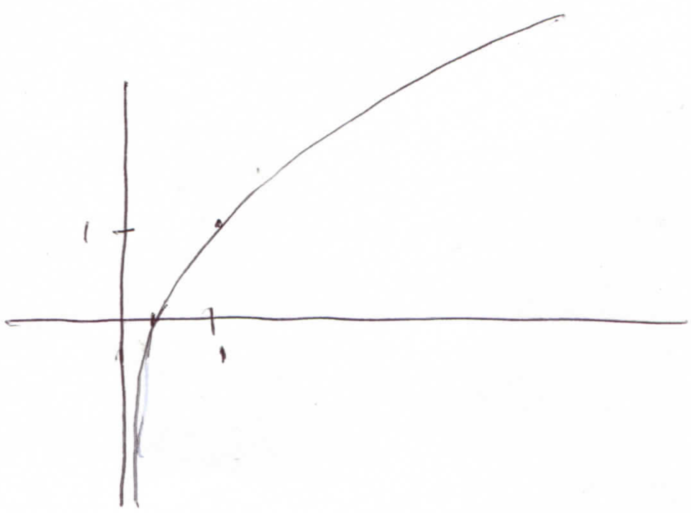
d.



e.



f.



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