Trig to the Point

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TRIG TO THE POINT

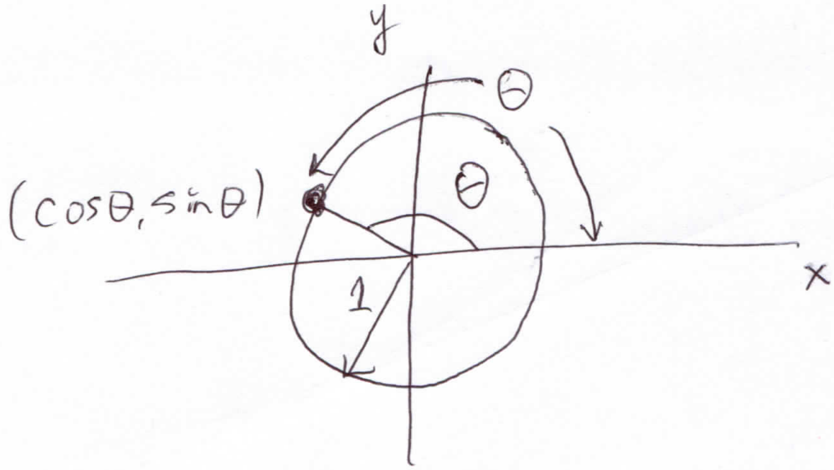
Trig (short for "trigonometry," shortening by popular demand) is often taught very poorly, culminating with a page filled with seemingly random formulas, to be brought to all exams, to minimize the random memorization that a confused instructor has asserted would otherwise be necessary. This is diametrically opposite to math (short for "mathematics") or any serious intellectual activity, where the goal is to have as small a list as possible of first principles, that is, things to be taken on faith, in pictures if possible, from which the entirety of the subject should follow logically and inevitably.

Trigonometry is a single picture, with a caption, as drawn below. Understanding trig only requires translating the details of this picture, which this very short monograph will do.

Much of this will appear in a more rigorous form in an upcoming book "Vectors to Trig to Geometry." You will see in the current small book that the definition and properties of trig functions in Chapter II do not require geometry; the larger book referred to, and Chapter III of this book, demonstrate how trig may be enhanced and applied much more extensively, including to a deeper understanding and application of geometry, with the basic parameters of geometry introduced first.

Chapters I and II require only one year of high-school algebra; Chapter III also requires some familiarity with geometry.

UNIT CIRCLE $x^2 + y^2 = 1$



$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$i^2 = (-1)$$

CHAPTER I: COMPLEX NUMBERS.

The easiest and most unified route to trig is via *complex numbers.*

Definitions 1.1. Begin with something we can't do. Since $x^2 \geq 0$ for any real x, the equation

$$x^2 = -1$$

has no real solution. Rather than retreating in dismay, we employ the brazen and surprisingly common and successful trick of giving a name to what (so far) doesn't exist.

By definition, "i" (short for "imaginary") is a number whose square is (-1):

$$i^2 = (-1) \quad \text{also denoted} \quad i \equiv \sqrt{(-1)}.$$

Imaginary numbers are real multiples of i; **complex numbers** are sums of real numbers and imaginary numbers $(x + iy)$, where x and y are real numbers:

$$\mathbf{C} \equiv \{\text{complex numbers}\} \equiv \{(x + iy) \mid x, y \text{ are real}\}.$$

The **real part** of $z \equiv (x + iy)$, denoted $\mathrm{Re}(z)$, is x; the **imaginary part**, denoted $\mathrm{Im}(z)$, is y.

We may add complex numbers and multiply complex numbers by real numbers:

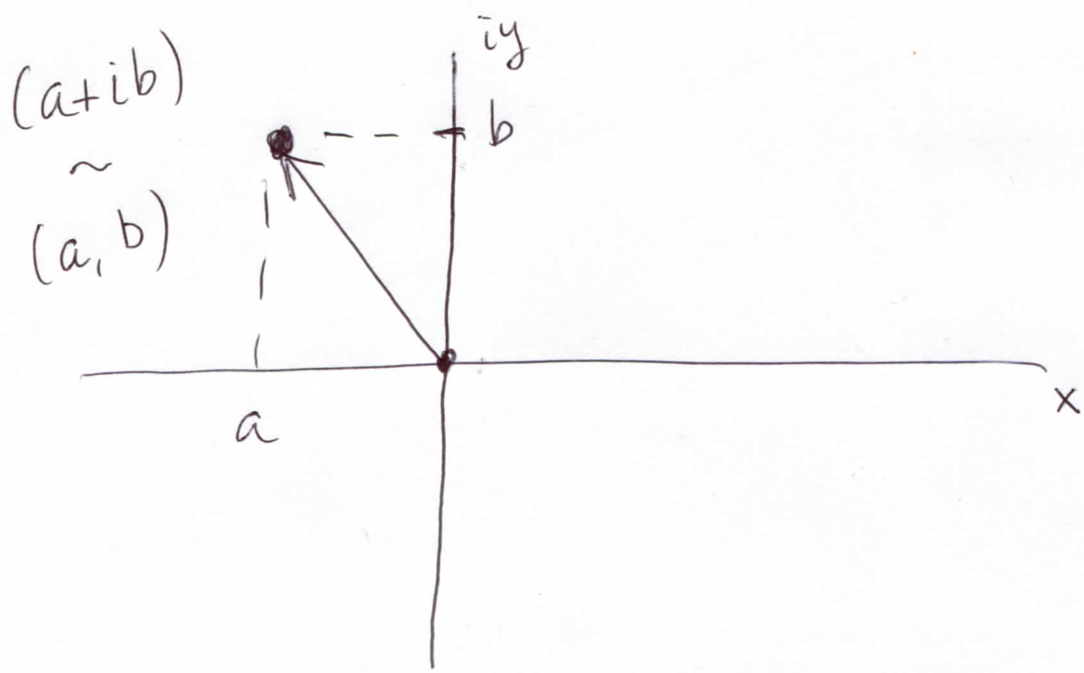
$$(a + ib) + (c + id) = (a + c) + i(b + d); \quad c(a + ib) = ca + i(cb). \quad (a, b, c, d \text{ real}).$$

For those who have seen vectors, these operations are not really new when we think of a complex number as a point or vector (in standard position) in the plane:

$$(a + ib) \sim (a, b) \quad \text{or} \quad < a, b >,$$

where "$\sim$" refers to not-necessarily-rigorous equating.

For example, $1 \sim (1, 0), i \sim (0, 1), (-1) \sim (-1, 0), (1 + 2i) \sim (1, 2)$, etc.



DRAWING 1.1

In this setting, the x-axis is called the **real axis R**, the y-axis is called the **imaginary axis iR**, and **C** is called the **complex plane**.

What *is* new is that we may multiply two complex numbers together or divide one complex number by another:

$$(a+ib)(c+id) = (ac+iad)+(icb+dbi^2) = (ac-db)+i(ad+cb) \quad (a,b,c,d \text{ real});$$

$$\frac{(a+ib)}{(c+id)} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i(bc-ad)}{(c^2+d^2)} = \left(\frac{(ac+bd)}{(c^2+d^2)}\right)+i\left(\frac{(bc-ad)}{(c^2+d^2)}\right) \quad (a,b,c,d \text{ real}, \ c^2+d^2 \neq 0).$$

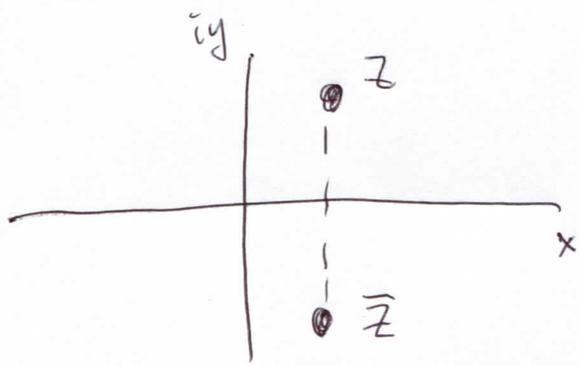
Definitions 1.2. Suppose a and b are real and $z \equiv (a+ib)$. The **conjugate** of z is

$$\overline{z} \equiv (a-ib).$$

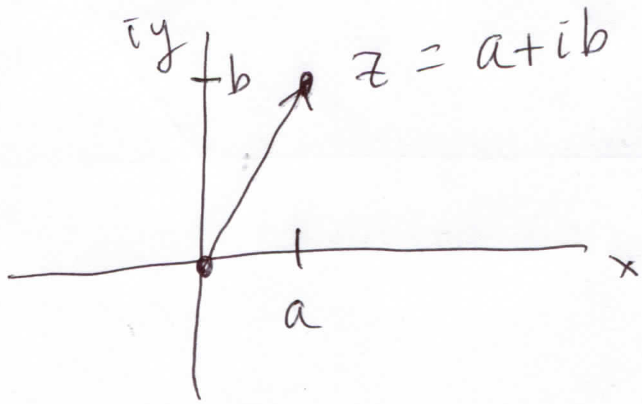
The **absolute value** of z is

$$|z| \equiv \sqrt{a^2+b^2}.$$

Note that $|z|$ is the length of the directed line segment representing the vector $< a, b >$ and conjugation is reflection through the x axis.



DRAWING 1.2



DRAWING 1.3

Examples 1.3. For $z \equiv (1+2i), w \equiv (2-i)$, get each of the following.

(a) $\overline{z}$. (b) $(z+3w)$. (c) zw. (d) $\frac{1}{w}$. (e) $\frac{w}{z}$. (f) $|z|$. (g) $|z-w|$. (h) $w\overline{w}$.

Solutions. (a) $(1-2i)$.

(b) $(1+2i)+(6-3i)=(7-i)$.

(c) $(1+2i)(2-i)=(1)(2-i)+(2i)(2-i)=(2-i)+(4i-2i^2)=(2-i)+(4i+2)=(4+3i)$.

(d) $\frac{1}{(2-i)}=\frac{(2+i)}{(2-i)(2+i)}=\frac{(2+i)}{5}=(\frac{2}{5}+i(\frac{1}{5})$.

(e) $\frac{(2-i)}{(1+2i)}=\frac{(2-i)(1-2i)}{(1+2i)(1-2i)}=\frac{(-5i)}{5}=-i$.

(f) $\sqrt{1^2+2^2}=\sqrt{5}$.

(g) $|-1+3i|=\sqrt{(-1)^2+3^2}=\sqrt{10}$.

(h) $(2-i)(2+i)=5$ (this is $|w|^2$).

We will now present a peculiar representation of complex numbers (Theorem 1.4(3)) that will simultaneously produce both of the desired trig functions sine and cosine in Chapter II.

Theorem 1.4. There exists a real number, denoted e (for "exponent" or "exponential"), and a family of complex numbers denoted $\{e^{i\theta}\}_{\theta \text{ real}}$, with the following properties.

(1) $\{e^{i\theta}\}_{\theta \text{ real}}$ has the exponential properties

$$(a)\ e^{i\theta}e^{i\psi}=e^{i(\theta+\psi)} \quad \text{and} \quad (b)\ \left(e^{i\theta}\right)^{\alpha}=e^{i\alpha\theta}$$

for all real θ, ψ, α.

(2) $|e^{i\theta}|=1$, for all real θ.

(3) Every nonzero complex number z has a **polar form**

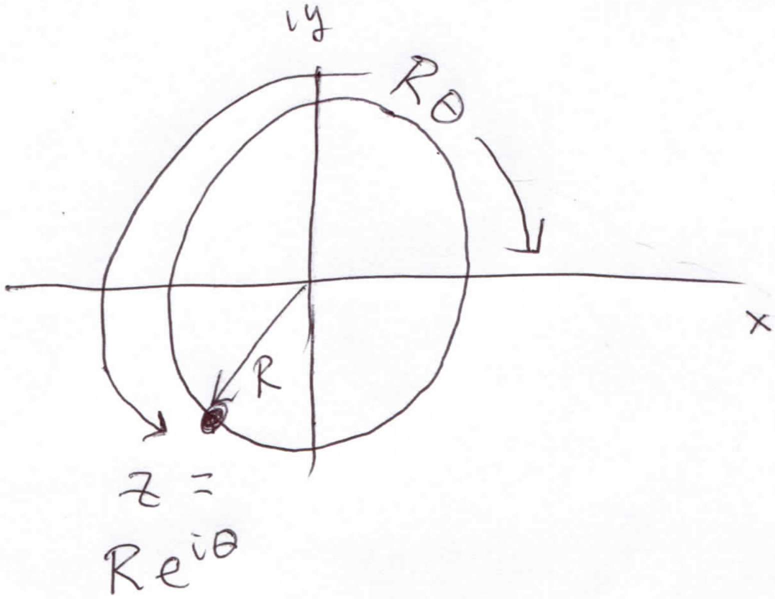
$$z=Re^{i\theta},$$

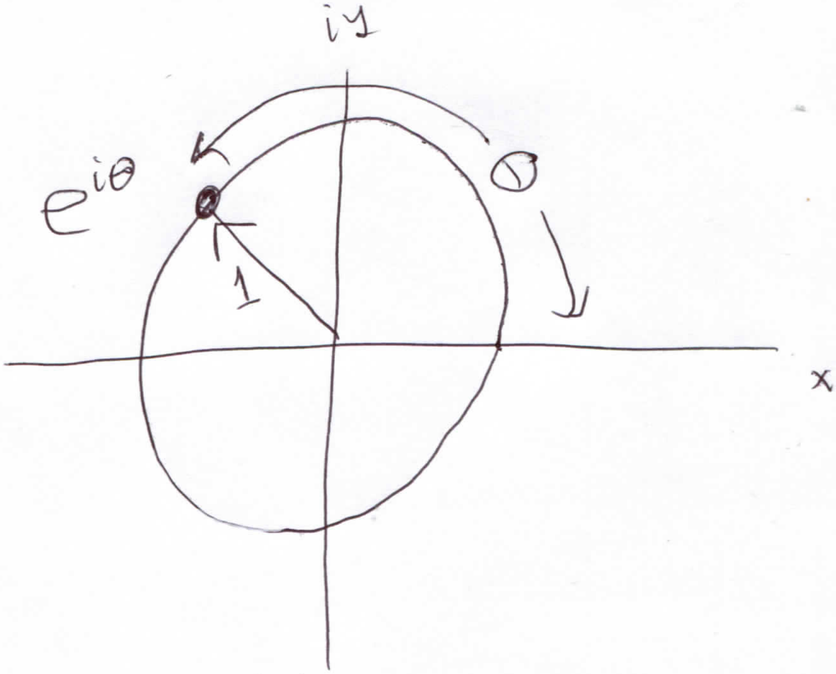
with $R=|z|, 0 \leq \theta < 2\pi$, and $R\theta$ equal to the length of $C_{\theta,R}$, where

$$C_{\theta,R} \equiv \{Re^{it} \,|\, 0 \leq t \leq \theta\}.$$

See Drawings 1.4 and 1.5. The **unit circle** $x^2+y^2=1$ in Drawing 1.5 is the special case $R=1$, from Theorem 1.4(3) and Drawing 1.4.

DRAWINGS 1.4 and 1.5





Terminology 1.5. As with positive real numbers, square roots of nonzero complex numbers come in pairs: if w and z are complex numbers, with

$$w^2 \equiv (w)(w) = z,$$

it is also true that

$$(-w)^2 = (-1)^2 w^2 = z;$$

that is, both w and $(-w)$ are square roots of z, and we must choose which square root is denoted $\sqrt{z}$ and which is denoted $(-\sqrt{z})$.

The polar form of Theorem 1.4(3) provides a simple way to distinguish the two square roots. If

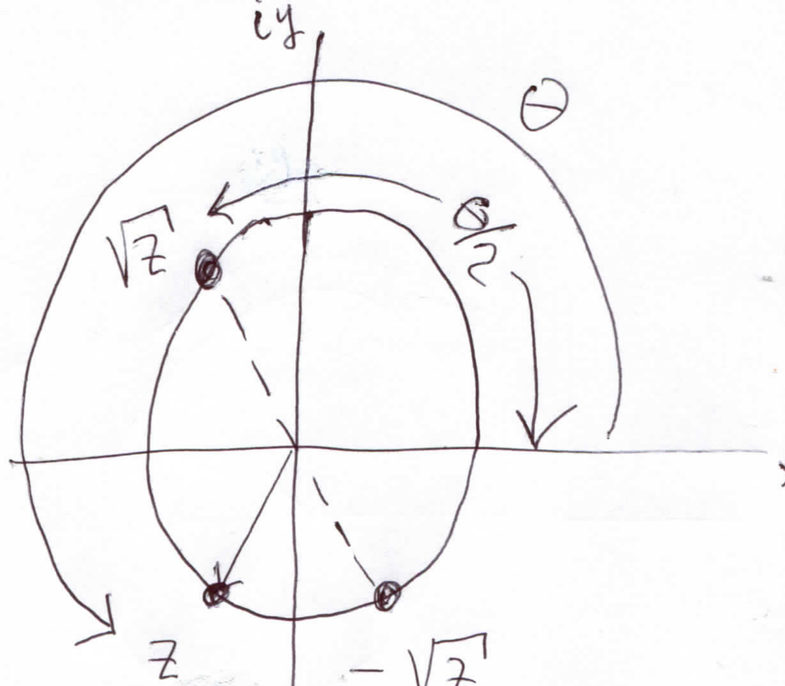
$$z = |z|e^{i\theta},$$

with $0 \le \theta < 2\pi$, then

$$\sqrt{z} \equiv \left(\sqrt{|z|}\right) e^{i\frac{\theta}{2}} \text{ and } (-\sqrt{z}) \equiv \left(-\sqrt{|z|}\right) e^{i\frac{\theta}{2}} = \left(\sqrt{|z|}\right) e^{i(\pi+\frac{\theta}{2})}.$$

Notice that $\sqrt{|z|}$ is already defined as the positive real number whose square is the positive real number $|z|$.

Notice also that our terminology is consistent with the $\sqrt{\cdot}$ terminology for positive real numbers



$$z = e^{i\theta} \rightarrow \sqrt{z} = e^{i\theta/2}, \quad -\sqrt{z} = e^{i(\pi+\frac{\theta}{2})}$$

DRAWING 1.6

Examples 1.6. Let's apply Theorem 1.4 for some particular choices of θ.

Since the circumference of the unit circle $x^2 + y^2 = 1$ is 2π, $e^{2\pi i} = 1$. We could then use the symmetry of the unit circle to get $e^{i\pi}, e^{i\frac{\pi}{2}}$, and $e^{i\frac{3\pi}{2}}$ (see Drawing 1.7; notice the equal spacing of $e^{i\frac{\pi}{2}}, e^{i\pi}$, etc.), or we could use the algebraic properties of Theorem 1.4, as follows.

Write $(a + ib) = e^{i\pi}$, for real a, b to be determined.

Since

$$1 = e^{2\pi i} = \left(e^{i\pi}\right)^2 = (a+ib)^2 = (a+ib)(a+ib) = a^2 + a(ib) + (ib)a + (ib)^2 = (a^2 - b^2) + i(2ab),$$

we have $1 = (a^2 - b^2)$ and $0 = 2ab$, so that either $a = 0$ or $b = 0$.

If $a = 0$, then $1 = -b^2$, which is impossible since b is real. Thus b must equal zero, so that $1 = a^2$ implies $a = \pm 1$. If a were positive 1, then $C_{\pi,1}$, from Theorem 1.4(3), would equal 2π, contradicting Theorem 1.4(3). Thus a must equal -1, so that

$$e^{i\pi} = -1.$$

Now write $(a + ib) = e^{i\frac{\pi}{2}}$, for some real a, b. As with $e^{i\pi}$, since

$$-1 = e^{\pi i} = \left(e^{i\frac{\pi}{2}}\right)^2 = (a+ib)^2 = (a^2 - b^2) + i(2ab),$$

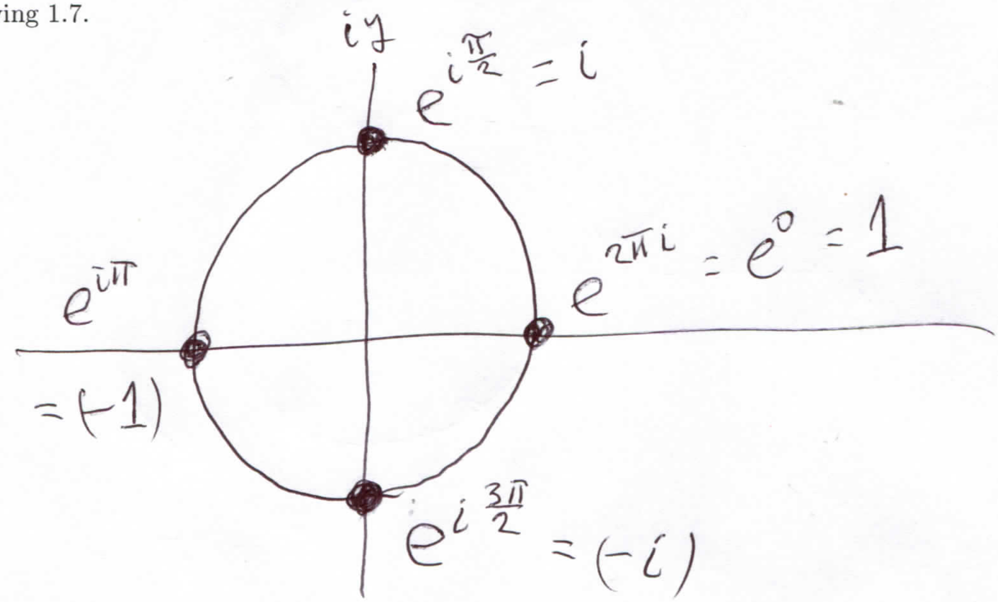
we have $-1 = (a^2 - b^2)$ and $0 = 2ab$; as with $e^{i\pi}$, a must equal zero and $b = \pm 1$; that is,

$$e^{i\frac{\pi}{2}} = \pm i; \qquad \text{the same argument shows that} \qquad e^{i\frac{3\pi}{2}} = \pm i.$$

Since the length of $C_{\frac{\pi}{2},1}$ is between 0 and π, $e^{i\frac{\pi}{2}}$ is in the upper half plane $y > 0$; thus we choose

$$e^{i\frac{\pi}{2}} = i; \qquad \text{similarly,} \qquad e^{i\frac{3\pi}{2}} = -i;$$

see Drawing 1.7.



DRAWING 1.7

Since the real numbers were enlarged by throwing in $i \equiv \sqrt{-1}$, it might appear (a famous science fiction writer asserted this) that the complex numbers could be enlarged by adding on $\sqrt{i}$. This is not true: both square roots of i are complex numbers.

We can calculate $\sqrt{i}$ directly. We want $a + bi$, with a and b real, so that

$$i = (a + bi)^2 = (a^2 - b^2) + i(2ba),$$

so that

$$(a^2 - b^2) = 0 \quad \text{and} \quad 2ba = 1,$$

which leads to $(a, b) = \pm\left[(\frac{1}{\sqrt{2}}) + i(\frac{1}{\sqrt{2}})\right]$.

Since $e^{i\frac{\pi}{2}} = i$, Theorem 1.4(3), as with the calculations of $e^{i\frac{\pi}{2}}$ and $e^{i\frac{3\pi}{2}}$, implies that

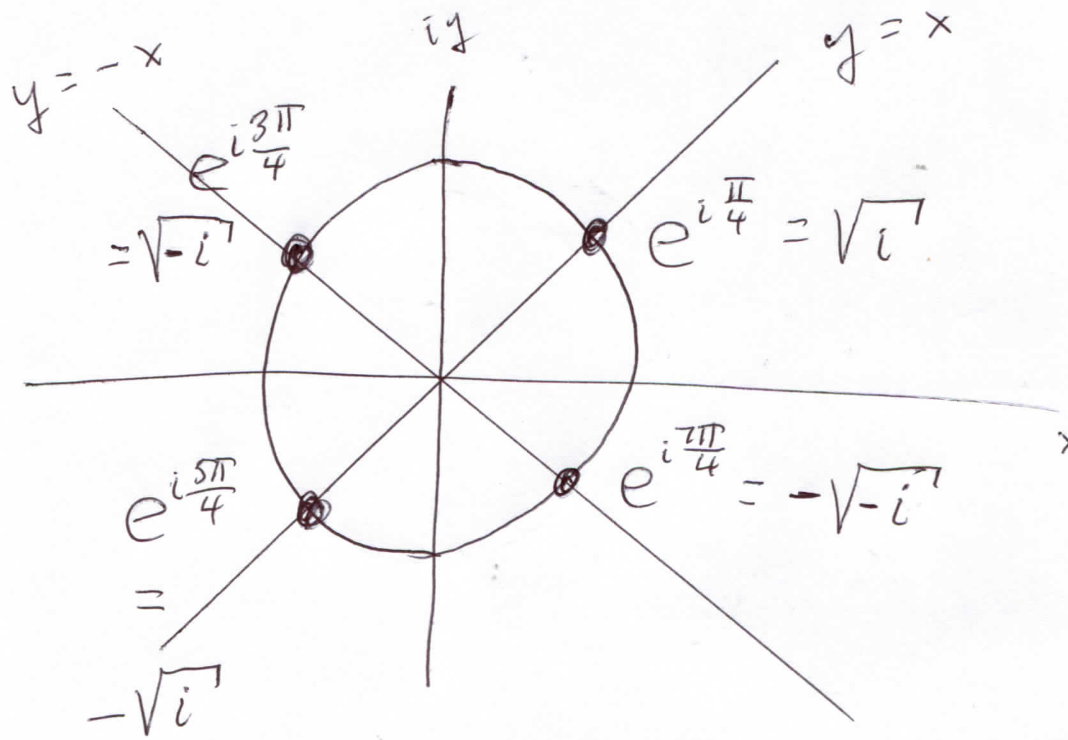
$$e^{i\frac{\pi}{4}} = \left[(\frac{1}{\sqrt{2}}) + i(\frac{1}{\sqrt{2}})\right] = \sqrt{i} \quad \text{and} \quad e^{i\frac{5\pi}{4}} = -\left[(\frac{1}{\sqrt{2}}) + i(\frac{1}{\sqrt{2}})\right] = -\sqrt{i}.$$

Note that $e^{i\frac{\pi}{4}}$ is on the line $y = x$, bisecting the first quadrant of the xy plane. See Drawing 1.8.

We leave it to the reader to similarly calculate

$$e^{i\frac{3\pi}{4}} = \left[-(\frac{1}{\sqrt{2}}) + i(\frac{1}{\sqrt{2}})\right] = \sqrt{-i} \quad \text{and} \quad e^{i\frac{7\pi}{4}} = \left[(\frac{1}{\sqrt{2}}) - i(\frac{1}{\sqrt{2}})\right] = -\sqrt{-i}.$$

Notice that $e^{i\frac{5\pi}{4}} = \overline{e^{i\frac{3\pi}{4}}}$, the conjugate of $e^{i\frac{3\pi}{4}}$, and $e^{i\frac{7\pi}{4}} = \overline{e^{i\frac{\pi}{4}}}$; you will also see reflections through the y axis in Drawing 1.8.



DRAWING 1.8

Remarks 1.7. Having traveled around the unit circle once, in going from $\theta = 0$ to $\theta = 2\pi$ (see Drawing 1.7), there is no reason we could not keep going as θ increases beyond 2π (see Drawing 1.9).

By Theorem 1.4,

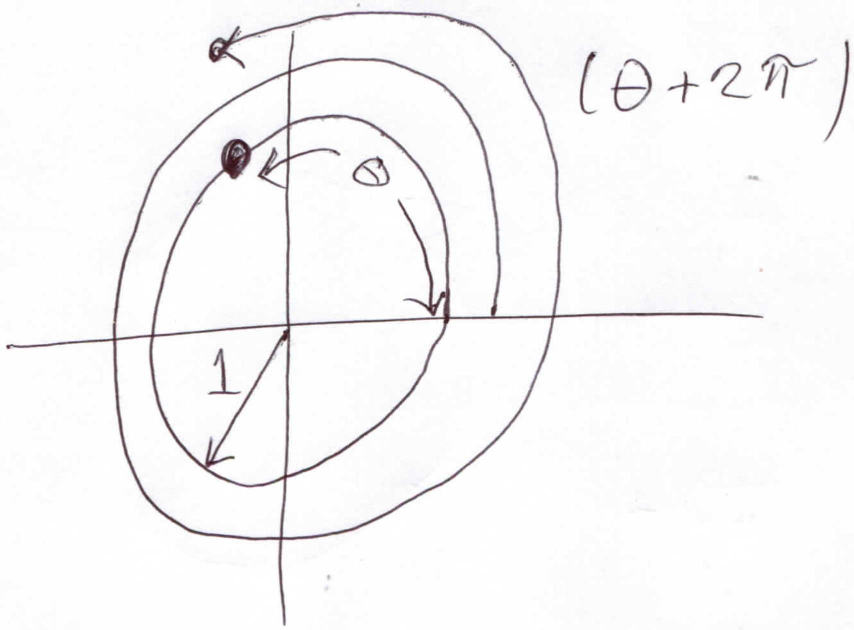
$$e^{i(2\pi+\theta)} = e^{2\pi i}e^{i\theta} = e^{i\theta}$$

for any real θ. As θ increases, $e^{i\theta}$ moves in the same direction around the unit circle $x^2 + y^2 = 1$; thus it is not surprising that we eventually come back to where we started. See Drawing 1.9.

The fact that $e^{i(\theta+2\pi)} = e^{i\theta}$ for any real θ implies that the arclength picture of Drawing 1.4 may be extended, as in Drawing 1.9, to values of θ not between 0 and 2π, if we allow arclength to accumulate even as we walk over previously traveled ground; for instance, we could let string steadily unravel behind us, and measure the length of string that has been released. Drawings 1.10 include a θ of $(4\pi + \frac{\pi}{2})$, traveling around the unit circle twice before settling at

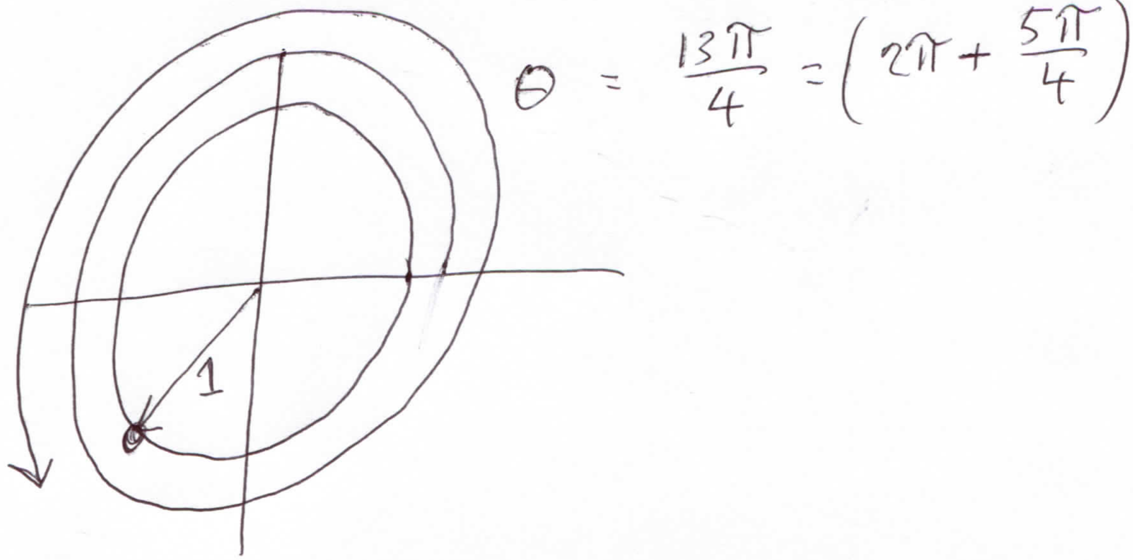
$$i = e^{i\frac{\pi}{2}} = e^{i(2\pi+\frac{\pi}{2})} = e^{i(4\pi+\frac{\pi}{2})} = \ldots.$$

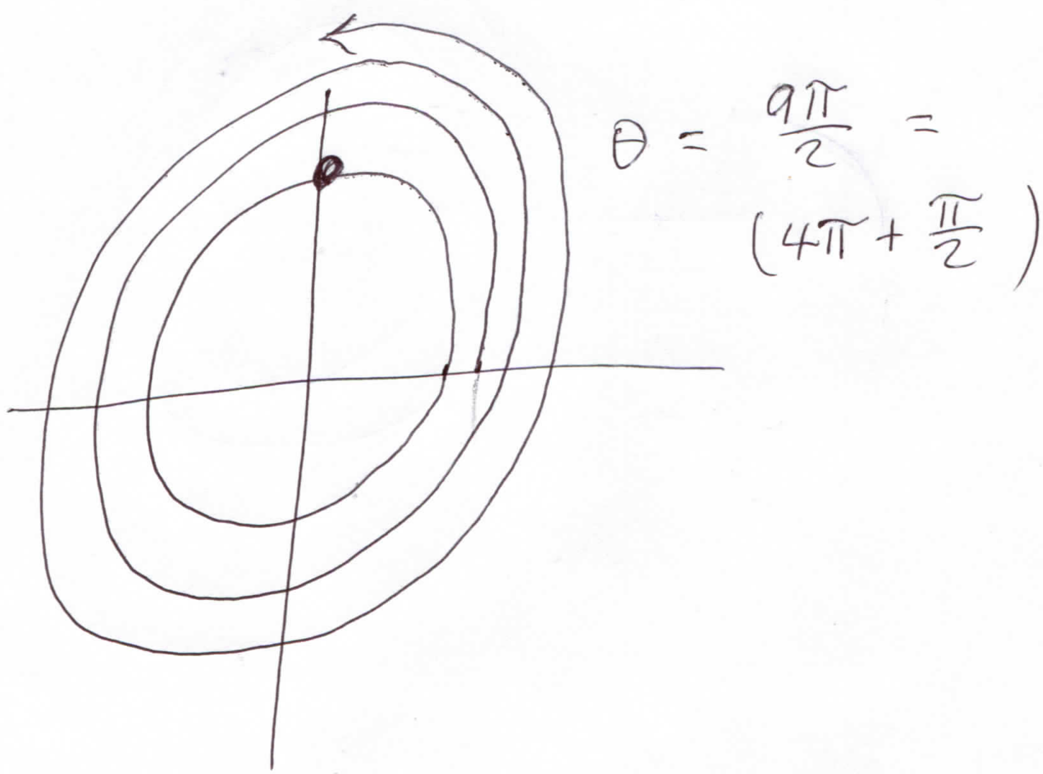
The direction of $e^{i\theta}$ traveling around the unit circle as θ increases is what determines what we call *counterclockwise* motion (see Theorem 1.4 and Drawings 1.5 and 1.7). The direction of travel of $e^{i\theta}$ as θ *decreases* is called *clockwise* motion, the opposite of counterclockwise. In Drawings 1.10 we've drawn $\theta = -\frac{3\pi}{4}$ and $-\frac{7\pi}{2}$.



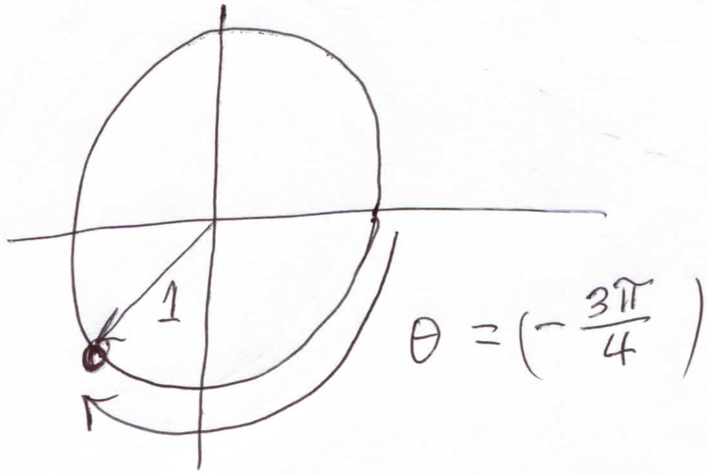
DRAWING 1.9

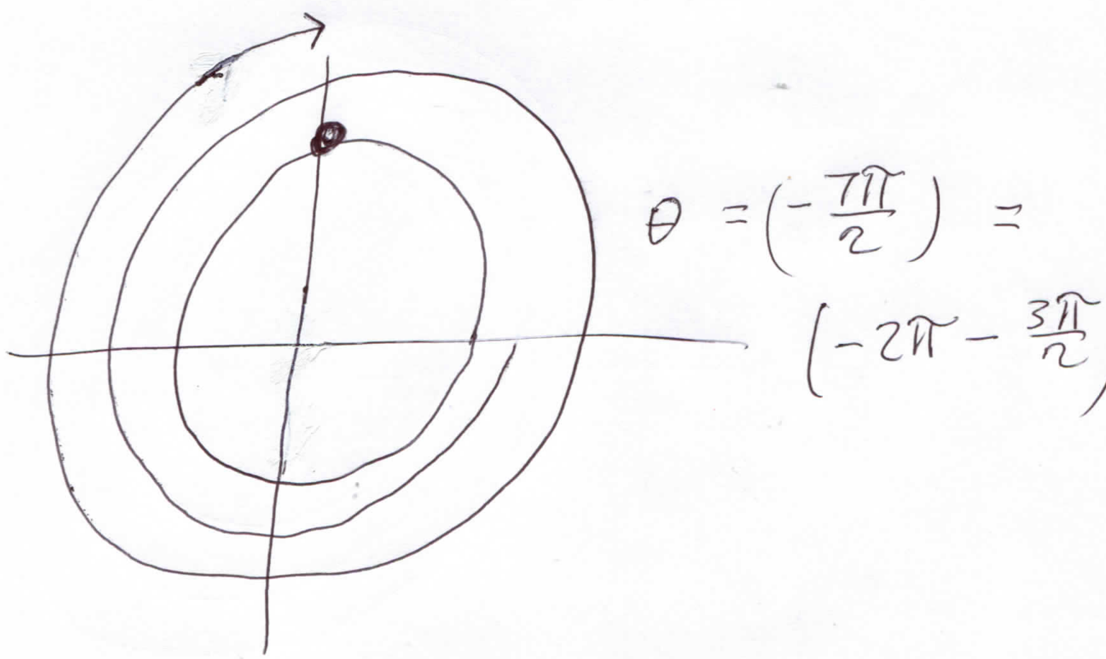
DRAWINGS 1.10





DRAWINGS 1.10 continued





CHAPTER II: EXPONENTIAL, COSINE, and SINE

The key results of this chapter are described by Drawing 2.1. Complex numbers (from Chapter I) will give a straightforward definition of the fundamental trig functions sine and cosine (Definition 2.1 and Drawing 2.1).

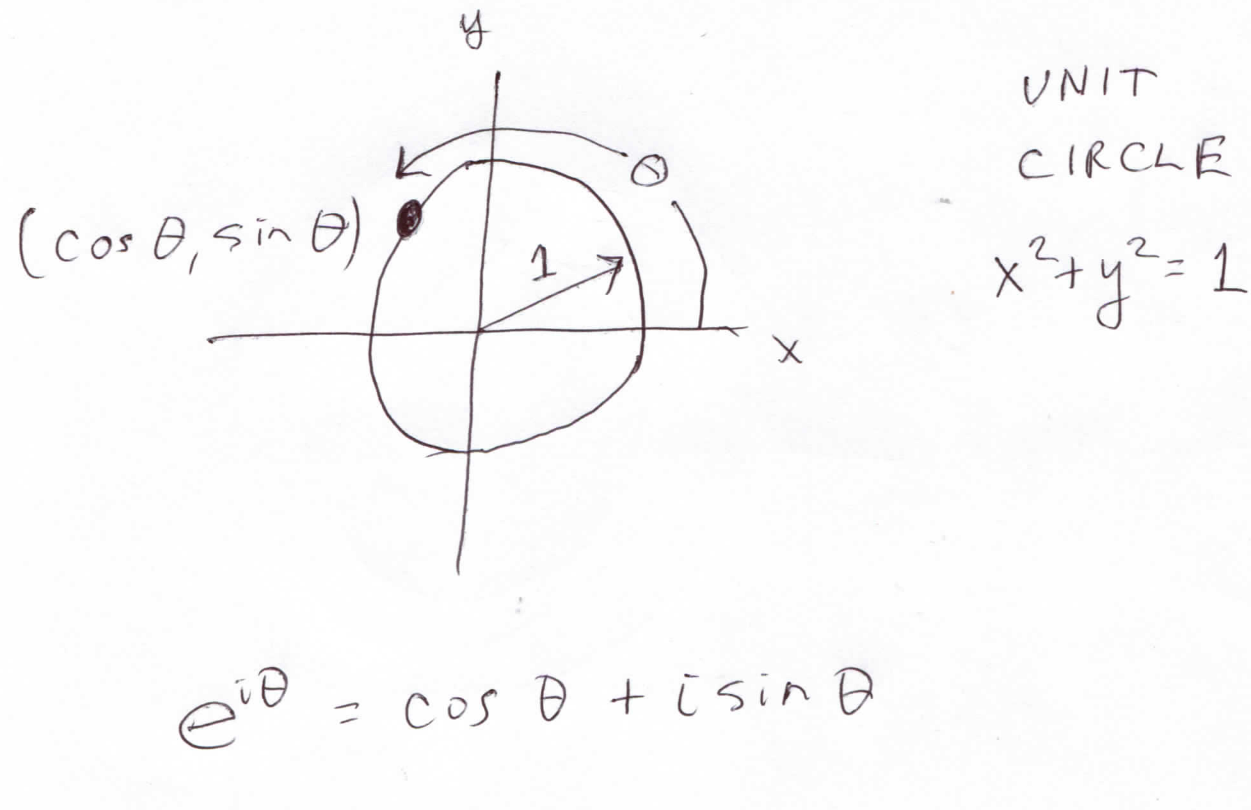
We must reiterate: *All you need to memorize about trig* is in the following definition and drawing (Drawing 2.1; compare to Drawing 1.5). Everything in trig follows from Drawing 2.1; Definition 2.1 is the *first principle* that any serious thinker tries to identify.

Definition 2.1. Let θ be any real number. Let "cos" be short for **cosine**, "sin" for **sine**. Define the **trigonometric functions**

$$\cos(\theta) \equiv \mathrm{Re}(e^{i\theta}), \quad \sin(\theta) \equiv \mathrm{Im}(e^{i\theta}).$$

Read out loud, "$\cos(\theta)$" reads "cosine of θ," "$\sin(\theta)$" reads "sine of θ."

Thus $(\cos\theta, \sin\theta)$ is the point on the unit circle $x^2+y^2=1$ such that the counterclockwise arclength from $(1,0)$ to $(\cos\theta, \sin\theta)$ is θ. See Drawings 2.1 and 1.5; arclength is generalized as in Remarks 1.7 and Drawings 1.10, when θ is greater than 2π or negative.



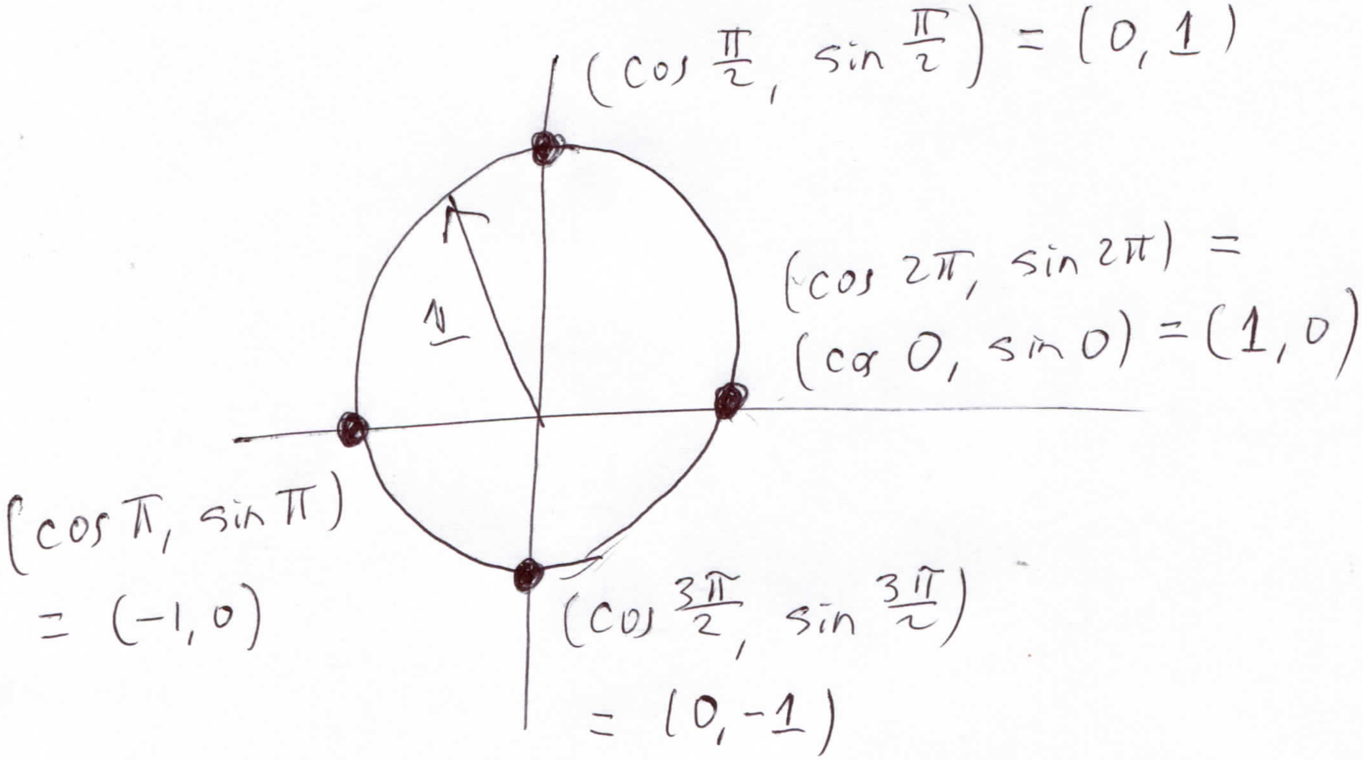
DRAWING 2.1

Definition 2.2. Stated explicitly as

$$e^{i\theta} = \cos\theta + i\sin\theta \ (\theta \text{ real}),$$

Definition 2.1 is called **Euler's formula**.

Here is Drawing 1.7 translated into the language of cosine and sine; in both Drawings, the symmetry of the unit circle and the popular definition of π as half the circumference of a circle of radius one, is sufficient to identify the appearance of $e^{i\theta}, \cos(\theta)$, and $\sin(\theta)$, for $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π.



DRAWING 2.2

Some Properties 2.3. From staring at the picture of $(\cos\theta, \sin\theta)$ in Drawing 2.1 and using the symmetry of the unit circle, the following properties seem believable, for any real θ (see Drawings 2.3; for (viii), see Drawing 1.9). Euler's formula provides straightforward proofs, but we particularly encourage the reader to derive these formulas, when needed, from drawings such as are in Drawings 2.3.

(i) $\cos(-\theta) = \cos\theta$.

(ii) $\sin(-\theta) = -\sin\theta$.

(iii) $\cos(\theta + \pi) = -\cos\theta$.

(iv) $\sin(\theta + \pi) = -\sin\theta$.

(v) $(\cos\theta)^2 + (\sin\theta)^2 = 1$.

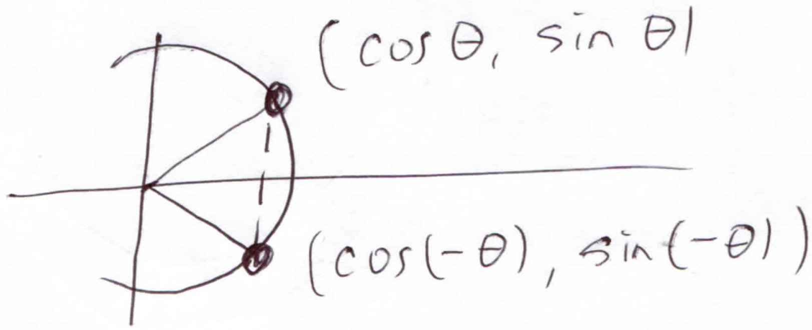
(vi) $|\sin\theta| \leq 1$ and $|\cos\theta| \leq 1$.

(vii) $\cos(\frac{\pi}{2} + \theta) = -\sin\theta = -\cos(\frac{\pi}{2} - \theta)$ and $\sin(\frac{\pi}{2} + \theta) = \cos\theta = \sin(\frac{\pi}{2} - \theta)$.

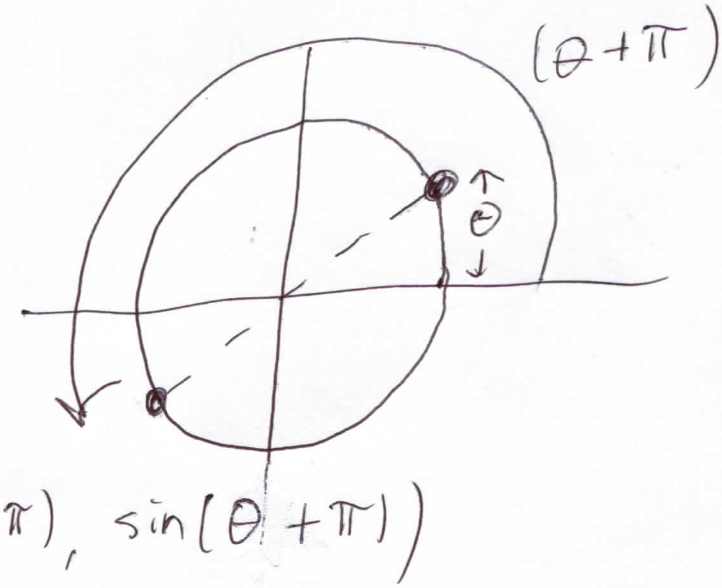
(viii) $\cos(\theta + 2k\pi) = \cos\theta$, $\sin(\theta + 2k\pi) = \sin\theta$, for any real θ, integer k (this is called **periodicity** of sine and cosine).

DRAWINGS 2.3

(i) & (ii)

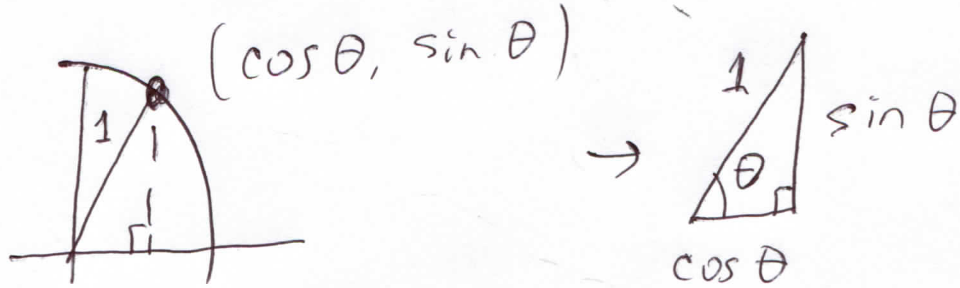


(iii) & (iv)

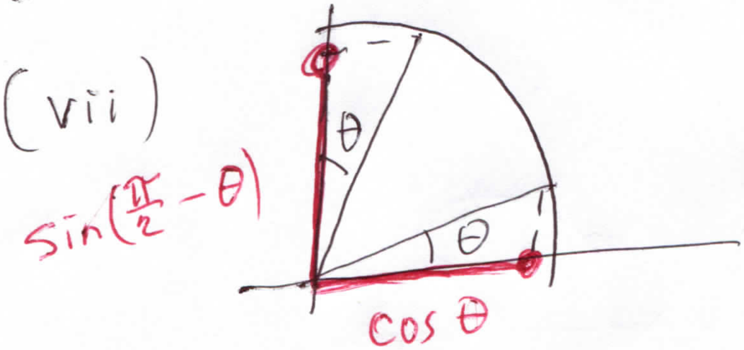


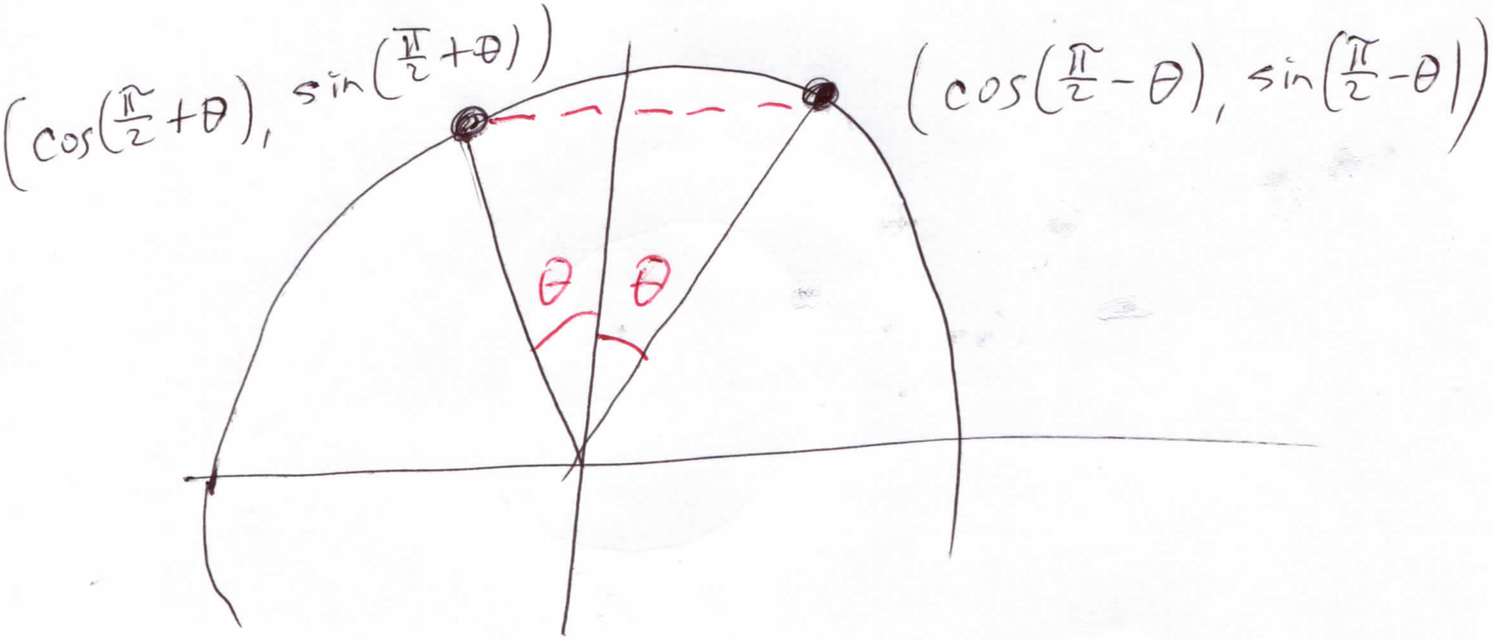
DRAWINGS 2.3 continued

(v) a (vi)



(vii)





(vii)

Proof of 2.3: See Theorem 1.4 for relevant properties of the exponential. We will make extensive use of Euler's formula in Definition 2.2.

The calculation

$$\cos(-\theta) + i\sin(-\theta) = e^{-i\theta} = e^{\overline{i\theta}} = \overline{e^{i\theta}} = \cos(\theta) - i\sin(\theta)$$

implies Properties 2.3(i) and (ii).

$$\cos(\theta+\pi) + i\sin(\theta+\pi) = e^{i(\theta+\pi)} = e^{i\theta}e^{i\pi} = e^{i\theta}(-1) = (-\cos(\theta)) + i(-\sin(\theta))$$

implies Properties 2.3(iii) and (iv).

Property 2.3(v) follows from Euler's formula, since $1 = |e^{i\theta}|^2$, by Theorem 1.4, while (vi) follows immediately from (v).

For (vii), make two calculations:

$$\cos(\frac{\pi}{2}-\theta)+i\sin(\frac{\pi}{2}-\theta) = e^{i(\frac{\pi}{2}-\theta)} = e^{i\frac{\pi}{2}}e^{-i\theta} = ie^{\overline{i\theta}} = i\overline{e^{i\theta}} = i(\overline{\cos\theta + i\sin\theta}) = i(\cos\theta - i\sin\theta) = \sin\theta + i\cos\theta$$

and

$$\cos(\frac{\pi}{2}+\theta) + i\sin(\frac{\pi}{2}+\theta) = e^{i(\frac{\pi}{2}+\theta)} = e^{i\frac{\pi}{2}}e^{i\theta} = i(\cos\theta + i\sin\theta) = (-\sin\theta) + i\cos\theta,$$

thus, equating real and imaginary parts gives the results.

For the periodicity (viii), write

$$\cos(\theta+2k\pi) + i\sin(\theta+2k\pi) = e^{i(\theta+2k\pi)} = e^{i\theta}e^{2k\pi i} = e^{i\theta}\left(e^{2\pi i}\right)^k = e^{i\theta}1^k = e^{i\theta} = \cos\theta + i\sin\theta,$$

as desired. □

We similarly get quick algebraic proofs of less-believable formulas for sine and cosine. These formulas are invaluable in many areas, including integration and Fourier series. Notice that 2.3(viii) follows from 2.4(i) and (ii).

Proposition 2.4. Let θ, ψ be arbitrary real numbers.

(i) $\cos(\theta+\psi) = \cos\theta\cos\psi - \sin\theta\sin\psi$.

(ii) $\sin(\theta+\psi) = \sin\theta\cos\psi + \sin\psi\cos\theta$.

(iii) $(\cos\theta)(\cos\psi) = \frac{1}{2}(\cos(\theta+\psi) + \cos(\theta-\psi))$.

(iv) $(\sin\theta)(\sin\psi) = \frac{1}{2}(\cos(\theta-\psi) - \cos(\theta+\psi))$.

(v) $(\sin\theta)(\cos\psi) = \frac{1}{2}(\sin(\theta+\psi) + \sin(\theta-\psi))$.

(vi) $(\cos\theta)^2 = \frac{1}{2}(1+\cos(2\theta))$.

(vii) $(\sin\theta)^2 = \frac{1}{2}(1-\cos(2\theta))$.

(viii) $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$.

(ix) $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$.

Proof of 2.4: For (i) and (ii), calculate

$$\begin{aligned}\cos(\theta+\psi) + i\sin(\theta+\psi) = e^{i(\theta+\psi)} &= e^{i\theta}e^{i\psi} = (\cos\theta + i\sin\theta)(\cos\psi + i\sin\psi)\\ &= (\cos\theta\cos\psi - \sin\theta\sin\psi) + i\left(\cos\theta\sin\psi + \sin\theta\cos\psi\right),\end{aligned}$$

so that equating the real and imaginary parts gives both *sum-of-angles* results simultaneously.

(iii) and (iv) follow from (i) and 2.3(i) and (ii); (v) follows from (ii) and 2.3(i) and (ii). (vi) and (vii) are (iii) and (iv) with $\theta = \psi$. (viii) and (ix) follow from Euler's formula. □

Examples 2.5. From Drawing 2.2,

$$\cos(0) = 1 = \cos(2\pi), \sin(0) = 0 = \sin(2\pi), \cos(\frac{\pi}{2}) = 0, \sin(\frac{\pi}{2}) = 1,$$

$$\cos(\pi) = -1, \sin(\pi) = 0, \cos(\frac{3\pi}{2}) = 0, \text{ and } \sin(\frac{3\pi}{2}) = -1.$$

From Examples 1.6, $e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(1+i)$, thus

$$\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} = \sin(\frac{\pi}{4}).$$

By 2.3(vii),

$$\cos(\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}}, \quad \sin(\frac{3\pi}{4}) = \frac{1}{\sqrt{2}}.$$

By 2.3(i) and (ii),

$$\cos(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}, \quad \sin(-\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}.$$

By 2.3(viii),

$$\cos(\frac{11\pi}{4}) = \cos(\frac{19\pi}{4}) = \cos(\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}}, \quad \sin(\frac{11\pi}{4}) = \sin(\frac{19\pi}{4}) = \sin(\frac{3\pi}{4}) = \frac{1}{\sqrt{2}}.$$

Sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ may be gotten simultaneously, as follows. Denote by

$$a \equiv \cos(\frac{\pi}{6}), \quad b \equiv \sin(\frac{\pi}{6}).$$

By 2.3(vii),

$$\cos(\frac{\pi}{3}) = b \text{ and } \sin(\frac{\pi}{3}) = a.$$

Thus, by Theorem 1.4,

$$(b + ia) = e^{i\frac{\pi}{3}} = \left(e^{i\frac{\pi}{6}}\right)^2 = (a+bi)^2 = (a^2 - b^2) + i(2ab),$$

implying $a = 2ab$, so that $b = \frac{1}{2}$; 2.3(v) now implies that $a = \frac{\sqrt{3}}{2}$, thus

$$\cos(\frac{\pi}{6}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2} \text{ and } \cos(\frac{\pi}{3}) = \sin(\frac{\pi}{6}) = \frac{1}{2}.$$

Finally, let's get $\cos(\frac{7\pi}{12})$ in three different ways.

From 2.4(vi):

$$\left(\cos(\frac{7\pi}{12})\right)^2 = \frac{1}{2}\left(1 + \cos(\frac{7\pi}{6})\right) = \frac{1}{2}\left(1 - \cos(\frac{\pi}{6})\right) \text{ (by 2.3(iii))} = \frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right).$$

This leaves us a choice of the negative or positive square root of $\frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right)$; since $\frac{\pi}{2} < \frac{7\pi}{12} < \frac{3\pi}{2}$, we know the cosine is negative; thus we choose

$$\cos(\frac{7\pi}{12}) = -\sqrt{\frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right)}.$$

From 2.4(i):

$$\cos(\frac{7\pi}{12}) = \cos(\frac{\pi}{4} + \frac{\pi}{3}) = \cos(\frac{\pi}{4})\cos(\frac{\pi}{3}) - \sin(\frac{\pi}{4})\sin(\frac{\pi}{3}) = (\frac{1}{\sqrt{2}})(\frac{1}{2}) - (\frac{1}{\sqrt{2}})(\frac{\sqrt{3}}{2}) = \frac{(1-\sqrt{3})}{2\sqrt{2}}.$$

From 2.4(i) again:

$$\cos(\frac{7\pi}{12}) = \cos(\frac{3\pi}{4} - \frac{\pi}{6}) = \cos(\frac{3\pi}{4})\cos(-\frac{\pi}{6}) - \sin(\frac{3\pi}{4})\sin(-\frac{\pi}{6}) = (-\cos(\frac{\pi}{4}))\cos(\frac{\pi}{6}) + \sin(\frac{\pi}{4})\sin(\frac{\pi}{6})$$

(by 2.3(vii), (i), and (ii)),

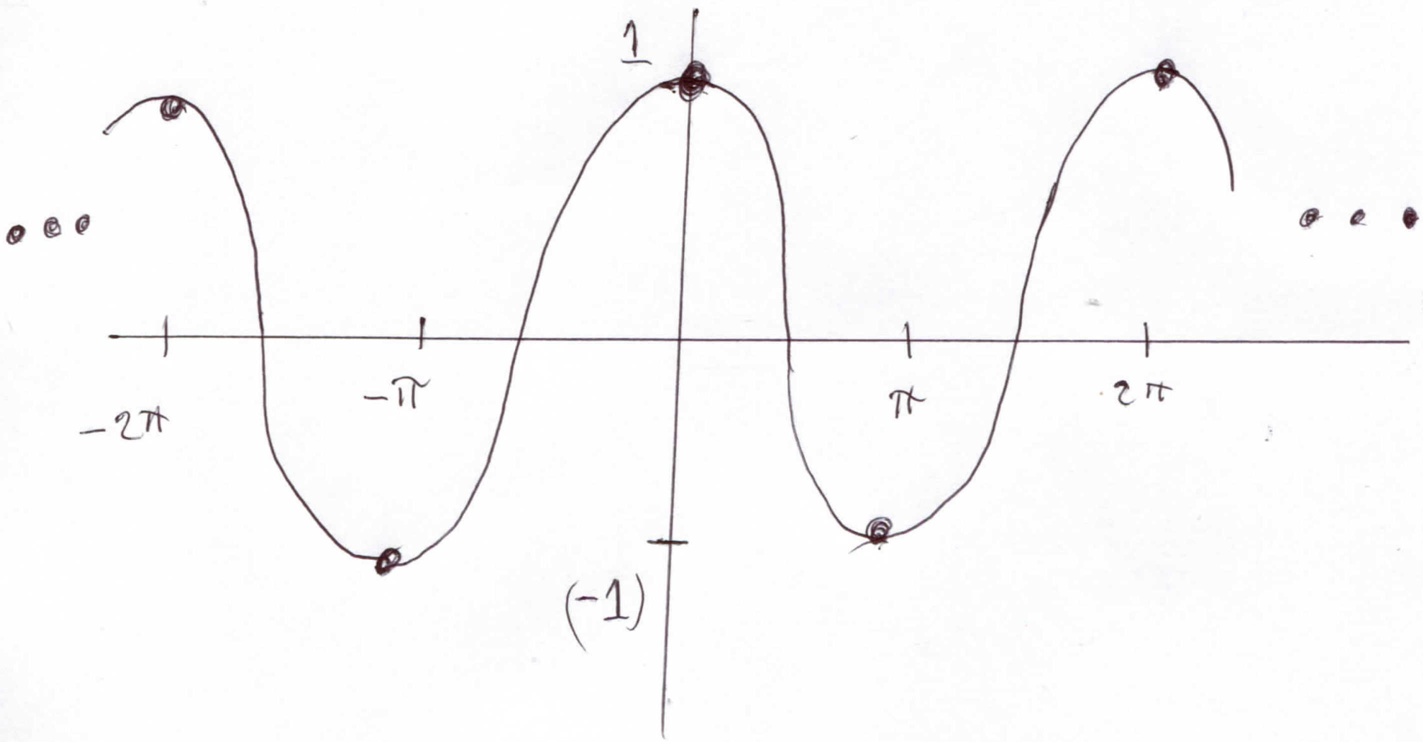
$$= (-\frac{1}{\sqrt{2}})(\frac{\sqrt{3}}{2}) + (\frac{1}{\sqrt{2}})(\frac{1}{2}) = \frac{(1-\sqrt{3})}{2\sqrt{2}}.$$

Remarks 2.6. The periodicity of cosine and sine (2.3(viii)) causes them to be good models of *waves*, such as sound waves and electromagnetic waves, including light and radio. See Drawing 2.4 below for cosine.

A general function describing waves is

$$f(x) \equiv A\cos(\gamma x - \psi),$$

where A and γ are positive real numbers and ψ is real. A is called **amplitude**, γ corresponds to **frequency**, and ψ is a phase shift. For example, if $f(x)$ is describing sound, A corresponds to volume and γ to pitch.



DRAWING 2.4

CHAPTER III: SOME GEOMETRICAL PERSPECTIVE

This chapter is aimed at readers who have seen some geometry; the previous chapters required no geometry.

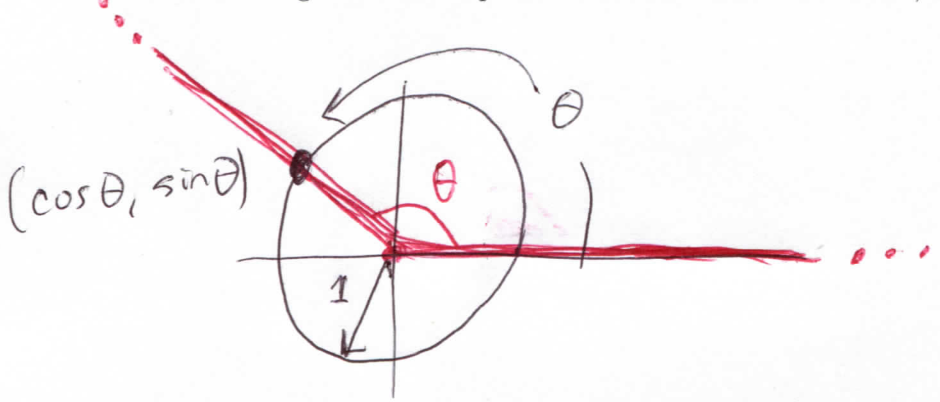
Below (Drawing 3.1) we reproduce Drawing 2.1, with two significant (from the point of view of geometry) rays drawn in red. The angle that θ is the measure of is one of the two determined by those rays. The units of θ are **radians**.

If we expand our picture of sine and cosine by a factor of $r > 0$, then both arclength and radius are multiplied by r, as in Drawing 3.2 below. Notice that the angle measure θ in radians is

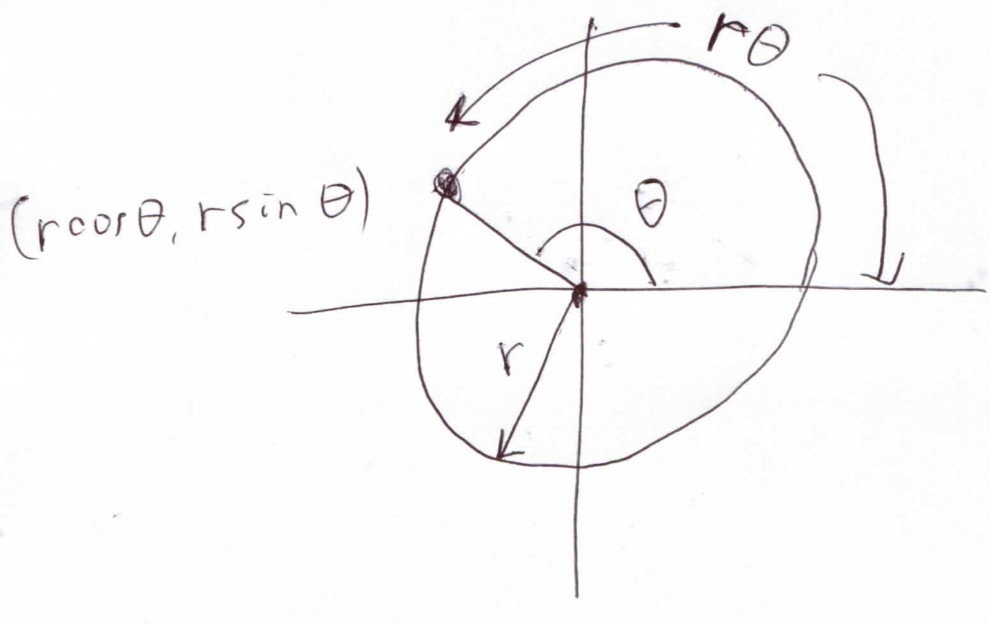
$$\theta = \frac{\text{arclength}}{\text{radius}};$$

thus radians are *unitless*, so long as radius and arclength are measured with the same units of length.

When we think of θ in Drawing 2.1 as the measure of an angle, trig may be applied to polygons, with devastating effect. See in particular the laws of cosine and sine, 3.5 and 3.6.



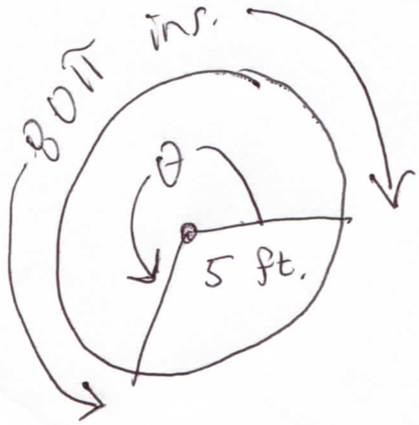
DRAWING 3.1



DRAWING 3.2

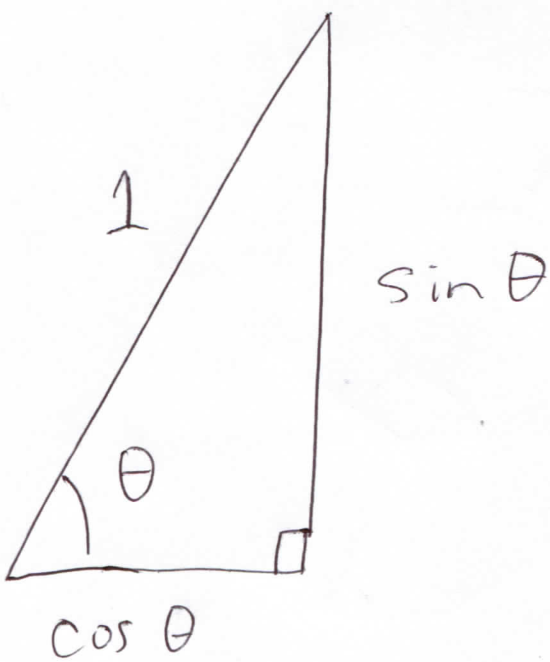
Example 3.1. In the picture (Drawing 3.3) below, θ, in radians, is

$$\frac{80\pi \text{ inches}}{5 \text{ feet}} = \frac{80\pi \text{ inches}}{60 \text{ inches}} = \frac{4\pi}{3}.$$



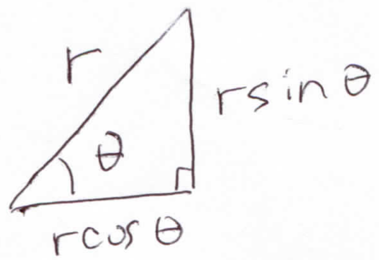
DRAWING 3.3

Cosine and Sine in Right Triangles 3.2. For $0 < \theta < \frac{\pi}{2}$, the picture of sine and cosine in Drawing 2.1, with a horizontal line of length $\cos\theta$ and a vertical line of length $\sin\theta$ added produce the following right triangle (Drawing 3.4; see Drawings 2.3(v) and (vi)).



DRAWING 3.4

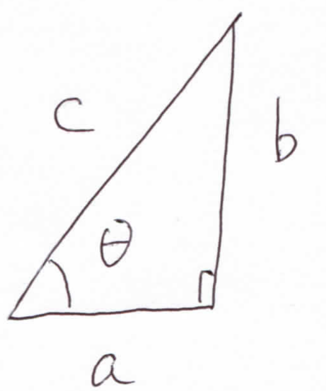
When we expand Drawing 3.4 by $r > 0$, as we did in Drawing 3.2, we get the following picture.



Notice now that cosine and sine are ratios of sides. More generally, for a right triangle with hypotenuse of length c and legs of lengths a and b, if θ is the measure of the angle formed by the hypotenuse and the leg of length a, then

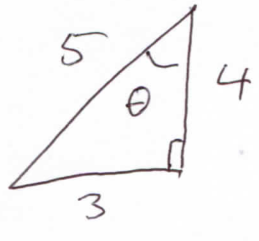
$$\cos\theta = \frac{a}{c} \text{ ("adjacent over hypotenuse") and } \sin\theta = \frac{b}{c} \text{ ("opposite over hypotenuse").}$$

See Drawing 3.5 below.



DRAWING 3.5

Example 3.3. Find $\cos\theta$ and $\sin\theta$ in the following right triangle (Drawing 3.6).



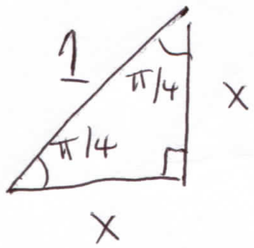
DRAWING 3.6

Solution. $\cos\theta = \frac{4}{5}$, $\sin(\theta) = \frac{3}{5}$.

Famous angles 3.4. We calculated sines and cosines of $\frac{\pi}{6}, \frac{\pi}{4}$, and $\frac{\pi}{3}$ in Examples 2.5, purely algebraically. Here we show how geometry and 3.2 can give us a more pictorial derivation.

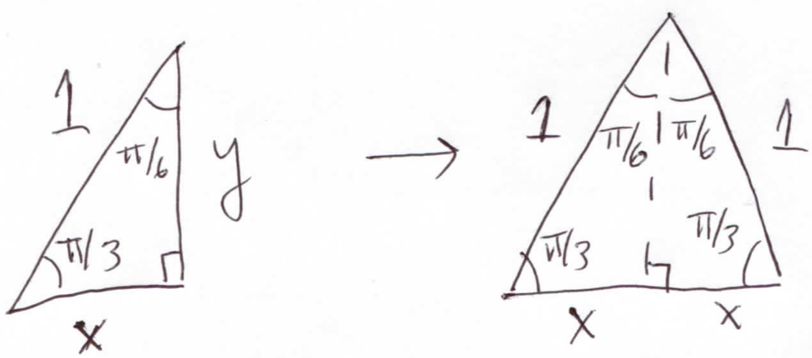
Besides the Pythagorean theorem, the only geometry we need is that, in a triangle, when angles have equal measure, the sides opposite those angles have equal measure; also the sum of all three angles is π radians.

Let $x \equiv \cos(\frac{\pi}{4})$. We have the following picture (Drawing 3.7), so that by the Pythagorean theorem, $2x^2 = 1$, so that $x = \frac{1}{\sqrt{2}}$.



DRAWING 3.7

Now let $x \equiv \sin(\frac{\pi}{6}), y \equiv \sin(\frac{\pi}{3})$. We have the following right triangle; paste a duplicate to its right side (Drawing 3.8).



DRAWING 3.8

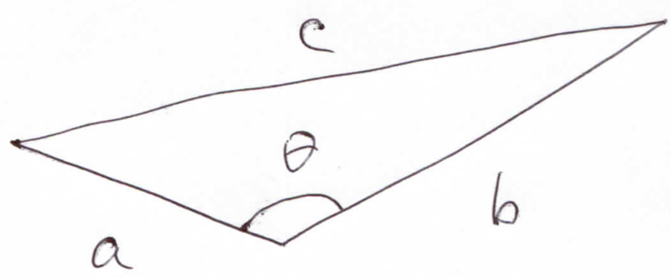
The larger triangle has all angles equal, hence it is equilateral; in particular, $x = \frac{1}{2}$, so that, by the Pythagorean theorem, $y = \frac{\sqrt{3}}{2}$.

The following are best derived with vectors, as in the reference from the first page.

Law of Cosines 3.5. If c is the length of the side of a triangle opposite the angle of measure θ, and a and b are the lengths of the other two sides, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

Notice that $\theta = \frac{\pi}{2}$ yields the Pythagorean theorem.

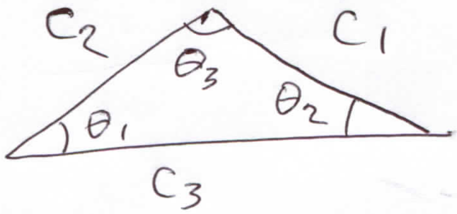


DRAWING 3.9

Law of Sines 3.6. If, in a triangle, c_j is the length of the side opposite the angle of measure $\theta_j, j = 1, 2, 3$, then

$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2} = \frac{\sin\theta_3}{c_3}.$$

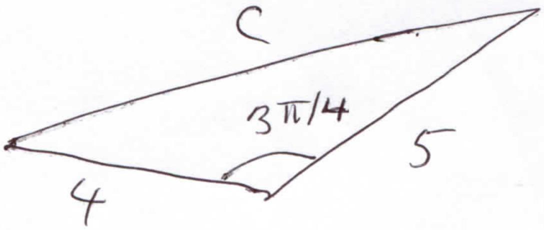
Notice that if $\theta_1 = \frac{\pi}{2}$ we get the formula for sine from 3.2.



DRAWING 3.10

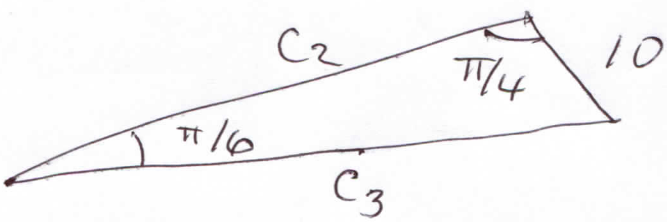
Examples 3.7. In each of the following, fill in all missing lengths of sides.

(a)



DRAWINGS 3.11

(b)



Solutions. (a) $c = \sqrt{4^2 + 5^2 - 2 \times 4 \times 5\cos(\frac{3\pi}{4})} = \sqrt{41 - 40(-\frac{1}{\sqrt{2}})} = \sqrt{41 + \frac{40}{\sqrt{2}}}$.

(b) The missing angle is $\pi - (\frac{\pi}{4} + \frac{\pi}{6}) = \frac{7\pi}{12}$, thus

$$c_2 = \left(\sin(\frac{7\pi}{12})\right)\frac{10}{\sin(\frac{\pi}{6})} = 20\left(\sin(\frac{7\pi}{12})\right), c_3 = \left(\sin(\frac{\pi}{4})\right)\frac{10}{\sin(\frac{\pi}{6})} = \frac{20}{\sqrt{2}};$$

we leave it to the reader to get $\left(\sin(\frac{7\pi}{12})\right)$ (see Examples 2.5).

Here is an older way of measuring angle.

Definition 3.8. For ψ a real number ψ **degrees** means $\psi\left(\frac{\pi}{180}\right)$ radians. Thus 2π radians is 360 degrees, π radians is 180 degrees, etc.

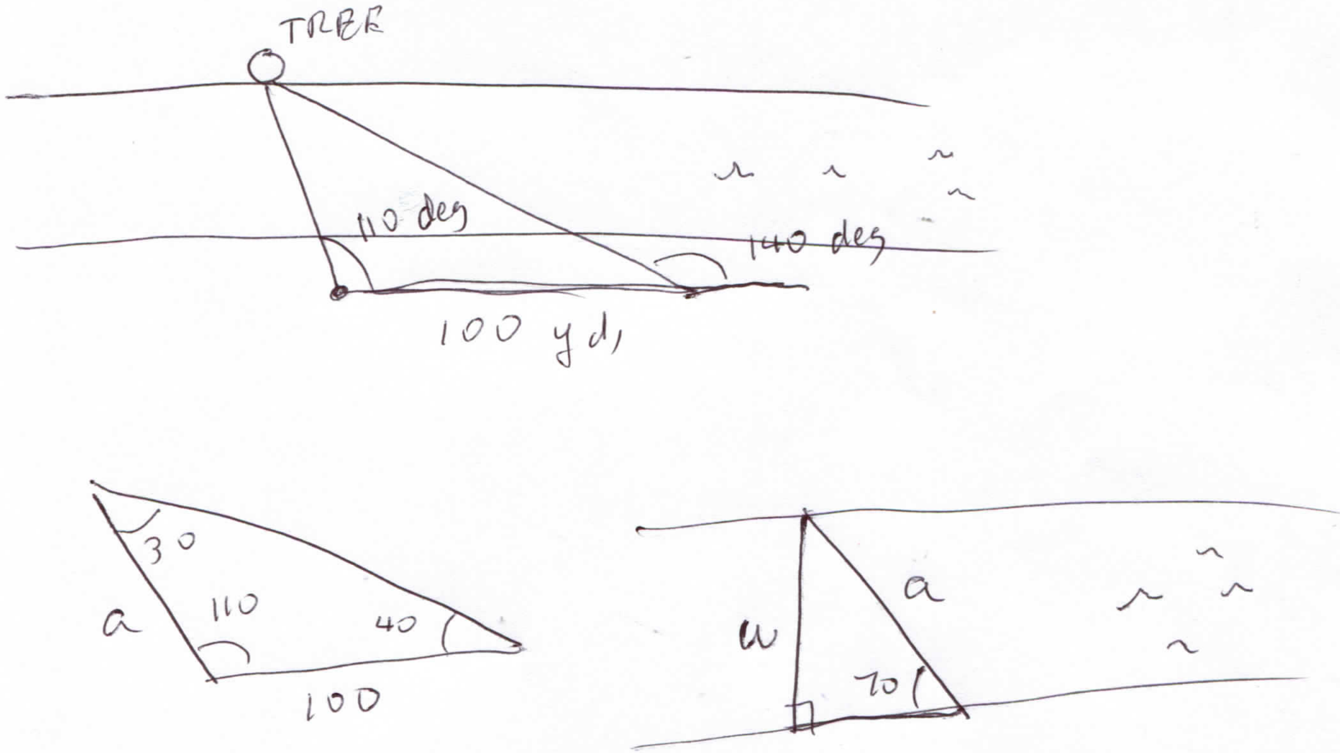
Degrees as angle measurement is believed to have originated with the Sumerians in about 2,000 BC; it's consistent with their predilection for multiples of 60, such as a base 60 number system.

Example 3.9. Suppose you are on one side of a straight river and you need to get to the other side. You can swim only a certain distance without drowning or you want to use a leaky boat that will travel only a certain distance before sinking; either way, you need to know the width of the river before you try to charge across.

You make two measurements, 100 yards apart, of the angle between your line of sight to a tree on the other side and the edge of the river, as drawn below. Use these to get the width of the river. Use a calculator to make decimal approximations of sines.

Solution. Use the Law of Sines to get the length labeled a: $\frac{a}{\sin(40\,\text{deg})} = \frac{100}{\sin(30\,\text{deg})} = 200$, so $a = 200\sin(40\,\text{deg})$. Now draw a right triangle with a as the hypotenuse length, to get w, the width of the river (see below): $w = a\sin(70\,\text{deg}) = 200\sin(40\,\text{deg})\sin(70\,\text{deg}) \sim 121$ yards.

Notice that we got all this information about the other side of the river without leaving our original side; it's a nice piece of detective work.



DRAWINGS 3.12

HOMEWORK

1. Suppose $z = (3 - 2i)$ and $w = (1 + i)$. Find each of the following

(a) zw (b) $z^2 \equiv z \cdot z$ (c) $\frac{w}{z}$ (d) $\overline{z}$ (e) $|w|^2$ (f) $|w - z|$.

2. Find $\sqrt{\sqrt{-i}}$. Describe it in two ways: as $e^{i\theta}$, for a specified real number θ; and as $(a + ib)$, for specified real a, b.

3. For each of the following θ, find $e^{i\theta}$; that is, write it as $(a + ib)$, for some real a, b.

(a) $\theta = 7\pi$.

(b) $\theta = \frac{11\pi}{2}$.

(c) $\theta = \frac{5\pi}{6}$.

4. Suppose $e^{i\theta} = (\frac{2\sqrt{2}}{3} - \frac{i}{3})$, for some real θ. Find $\cos(\theta)$ and $\sin(\theta)$.

5. Find $\sin(\frac{7\pi}{12})$.

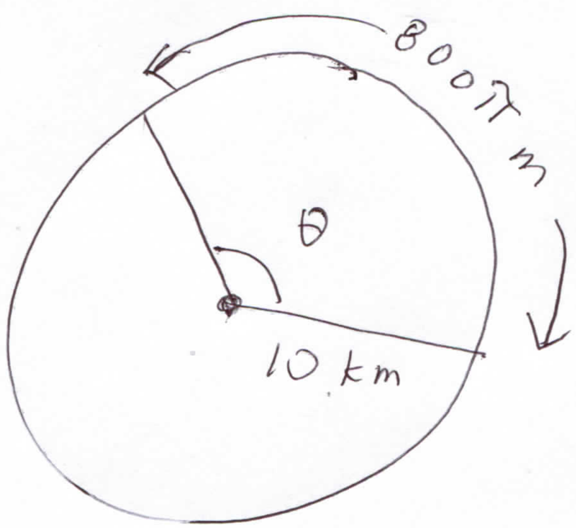
6. Get two different-looking expressions for each of sine and cosine of $\frac{5\pi}{12}$, by using different parts of 2.3 and 2.4 (see $\cos(\frac{7\pi}{12})$ in Examples 2.5).

7. For each of the following θ, get sine and cosine of θ.

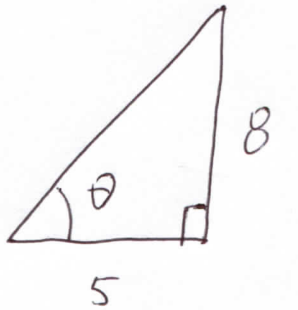
(a) $\theta = -\frac{2\pi}{3}$.

(b) $\theta = \frac{19\pi}{6}$.

8. Find θ, the measure in radians as drawn below.

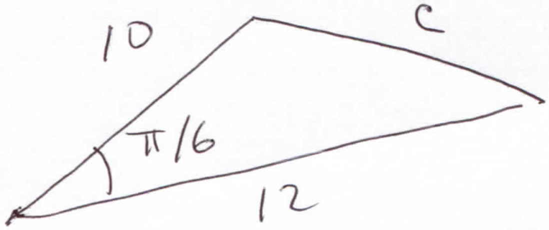


9. Find $\cos(\theta)$ and $\sin(\theta)$ in the picture below.

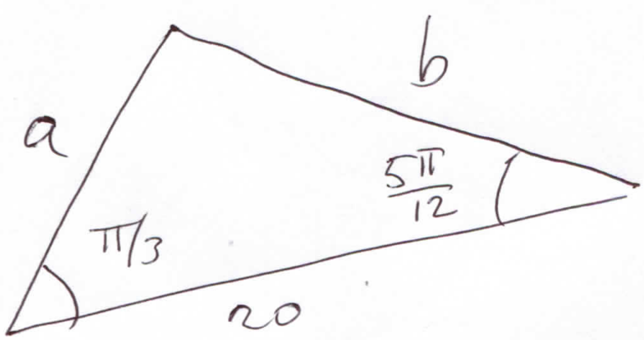


10. In each of the following, fill in all missing lengths of sides.

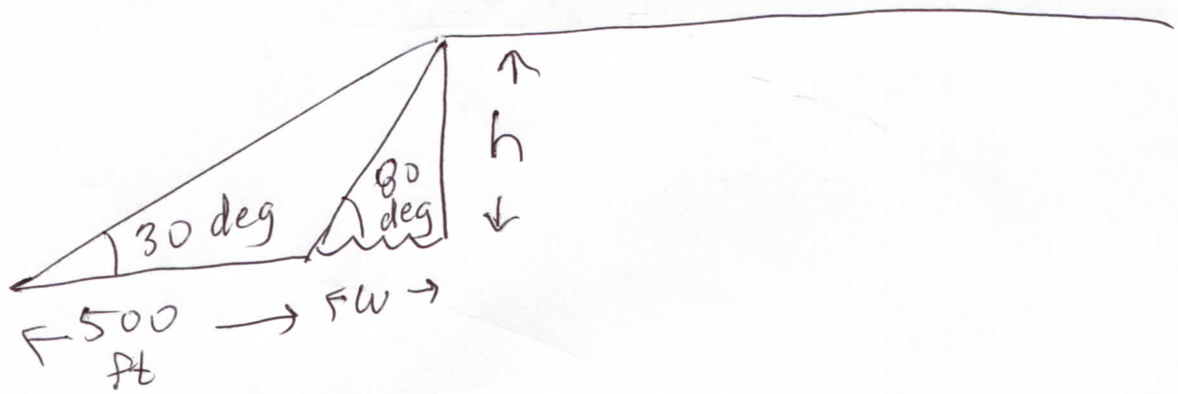
(a)



(b)



11. The drawing below shows a cliff of unknown height h and a moat of acid around the cliff of unknown width w. You measure the angle of elevation to the top of the cliff, at the edge of the moat and 500 feet from the edge of the moat. See drawing below. Use these measurements to get h and w. Use a calculator to get decimal approximations of sines and cosines. All angles are in degrees.



APPENDIX: What is e?

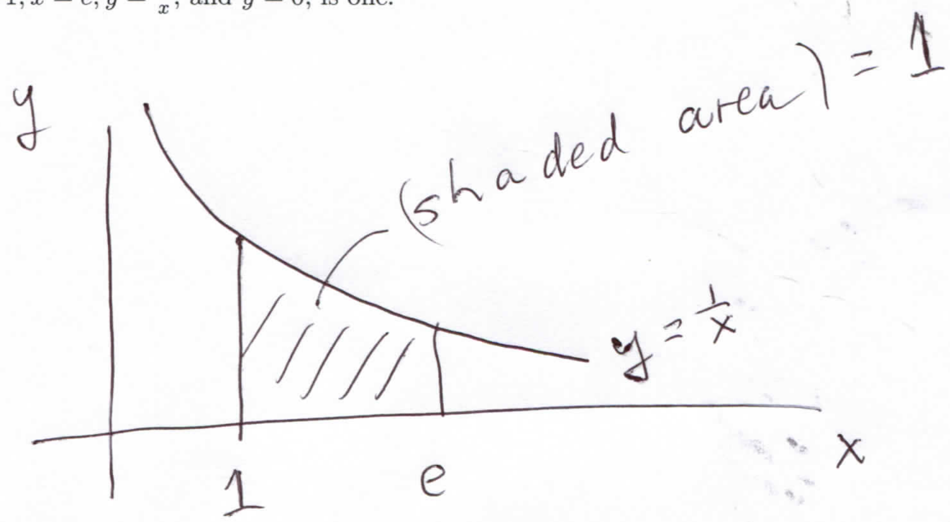
The number e is the limit $\lim_{n\to\infty}(1+\frac{1}{n})^n$. This means that the entries in the right-hand column below may be forced to be as close as you want to e, by making n large enough.

n	$(1+\frac{1}{n})^n$
1	$2 = (1+\frac{1}{1})^1$
2	$2.25 = (1+\frac{1}{2})^2$
5	$2.48832 = (1+\frac{1}{5})^5$
10	$2.59374246 = (1+\frac{1}{10})^{10}$
20	$2.653297705 \sim (1+\frac{1}{20})^{20}$
100	$2.704813829 \sim (1+\frac{1}{100})^{100}$
$1,000$	$2.716923932 \sim (1+\frac{1}{1,000})^{1,000}$

e also equals the infinite sum $(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots)$, which by definition is the limit of finite sums $\lim_{n\to\infty}(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!})$, with the same interpretation about the entries in the right-hand column below as with the first table above.

n	$(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!})$
1	$2 = (1+\frac{1}{1!})$
2	$2.5 = (1+\frac{1}{1!}+\frac{1}{2!})$
3	$2.666... = (1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!})$
4	$2.708333.... = (1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!})$
5	$2.71666... = (1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!})$
10	$2.718281801 \sim (1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots+\frac{1}{10!})$

Here is yet another equivalent way to define e. e is a number such that the area between $x = 1, x = e, y = \frac{1}{x}$, and $y = 0$, is one.



One more equivalent definition, for those familiar with the term "(instantaneous) rate of change." It can be shown that, for any constant $b > 0$, the rate of change of $y = b^x$ is proportional to b^x; that is, there's a constant R_b so that, for any real x, the rate of change of b^x equals $R_b b^x$.
e is the number such that $R_e = 1$; that is, for any real x, the rate of change of e^x equals e^x.